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CONTENTS.

VOL. VII.

The Uses of a Line-Divider. By Miss Sarah Marks	Page 1
On an Integrating Anemometer. By Walter Baily, M.A.	7
On the Quadrant Electrometer. By J. Hopkinson, D.Sc., F.R.S.	7
Notes on the Use of Nicol's Prism. By James C. McConnel, B.A., Assistant Demonstrator at the Cavendish Laboratory, Cambridge	22
On some new Forms of Polarizing-Prism. By Mr. H. G. Madan, M.A., Eton College	39
The most Economical Potential-difference to employ with Incan- descent Lamps. By Professors W. E. Ayrton, F.R.S., and John Perry, M.E.	40
On the Second Law of Thermodynamics. By Mr. J. Macfarlane Gray	51
Lecture-Experiments on Spectrum Analysis. By E. Clemin- shaw, M.A., F.C.S.	51
On the Characteristic Curves and Surfaces of Incandescence Lamps. By J. A. Fleming, M.A., D.Sc. (Lond.), Fellow of St. John's College, Cambridge	55
On the Structure of Mechanical Models illustrating some Pro- perties of the Æther. By Prof. George Francis Fitzgerald, F.R.S.	74
On Binocular Glasses adjustable to Eyes having unequal Focal Lengths. By Colonel Malcolm, R.E., C.B.	80
Summary of Lecture on Calculating Machines, delivered before the Physical Society of London, March 28, 1885, by Joseph Edmonson, of Halifax	81
On a Chrono-barometer and Chrono-thermometer. By W. Ford Stanley	85

	Page
On the Theory of Illumination in a Fog. By Lord Rayleigh, F.R.S.	87
A Monochromatic Telescope, with Application to Photometry. By Lord Rayleigh, F.R.S.	90
On the Self-Regulation of the Compound Dynamo. By A. W. Rücker, M.A., F.R.S.	92
On the Determination of the Heat-Capacity of a Thermometer. By J. W. Clark, Assistant Professor of Physics in University College, Liverpool	113
On certain Cases of Electrolytic Decomposition. By J. W. Clark, Assistant Professor of Physics in University College, Liverpool	118
On the Sensitiveness of Selenium to Light, and the Development of a similar Property in Sulphur. By Shelford Bidwell, M.A., LL.B.	129
On the Error involved in Professor Quincke's Method of Calculating Surface-Tensions from the dimensions of Flat Drops and Bubbles. By A. M. Worthington, M.A., Clifton, Bristol.	145
On the Use of Daniell's Cell as a Standard of Electromotive Force. By J. A. Fleming, M.A., D.Sc. (Lond.), Fellow of St. John's College, Cambridge, Professor of Electrical Technology in University College, London. (Plate I.)	161
On Molecular Shadows in Incandescence Lamps. By J. A. Fleming, M.A., D.Sc. (Lond.), Fellow of St. John's College, Cambridge, Professor of Electrical Technology in University College, London	178
The Production of Monochromatic Light, or of a Mixture of Colours, on the Screen. By Capt. W. de W. Abney, R.E., F.R.S.	181
On the Thermoelectric Position of Carbon. By John Buchanan. (Plate II.)	185
On the Specific Refraction and Dispersion of Light by the Alums. By J. H. Gladstone, Ph.D., F.R.S.	194
A Comparison of the Standard Resistance-coils of the British Association with Mercury Standards constructed by M. J. R. Benoit of Paris and Herr Strecker of Würzburg. By R. T. Glazebrook, M.A., F.R.S., Fellow and Assistant Tutor of Trinity College, Cambridge	201

CONTENTS.

v

	Page
Problems on the Distribution of Electric Currents in Networks of Conductors treated by the Method of Maxwell. By J. A. Fleming, M.A., D.Sc. (Lond.), Fellow of St. John's College, Cambridge, Professor of Electrical Technology in University College, London. (Plates III. & IV.).....	215
On the Generation of Electric Currents by Sulphur Cells. By Shelford Bidwell, M.A., LL.B.	256
On the Law of the Electromagnet, and the Law of the Dynamo. By Prof. Silvanus P. Thompson, D.Sc., B.A.	265
A Note on the Electromotive Force of certain Tin Cells. By E. F. Herroun, of King's College, London	275
Note on the Verification of Thermometers at the Freezing-Point of Mercury. By G. W. Whipple, B.Sc., F.R.A.S., Superintendent of the Kew Observatory	283
Note on the Calibration of Galvanometers by a Constant Current. By T. Mather, Assistant in the Physical Department, City and Guilds of London Central Institution	285
Some Thermodynamical Relations.—Part I. By William Ramsay, Ph.D., and Sydney Young, D.Sc.	289
Some Thermodynamical Relations.—Part II. By William Ramsay, Ph.D., and Sydney Young, D.Sc.	307
Some Thermodynamical Relations.—Part III. By William Ramsay, Ph.D., and Sydney Young, D.Sc.	327
The Winding of Voltmeters. By Professors W. E. Ayrton, F.R.S., and John Perry, F.R.S.	334
On a Machine for Solving Equations. By C. V. Boys. (Plate V.)	355
On a Mechanical Method of Solving Quadratic and Cubic Equations, whether the Roots be real or impossible. By Henry Cunynghame, Barrister-at-Law	360
Note on some Organic Substances of High Refractive Power. By H. G. Madan, M.A., Eton College	364
Note on the Determination of the Volume of Mercury in a Thermometer. By A. W. Clayden, M.A., F.G.S., Bath College	367

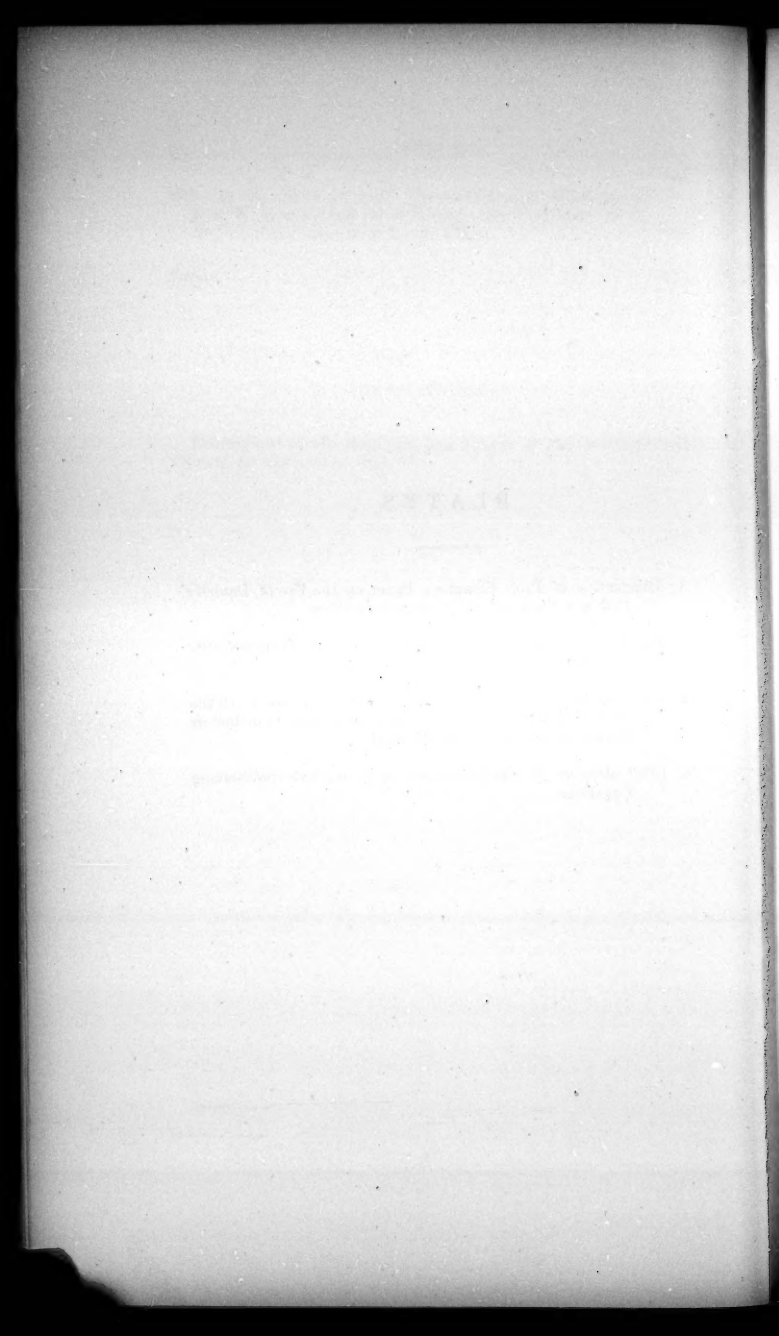
	Page
Note on the Paper on some Thermodynamical Relations by Prof. W. Ramsay and Dr. S. Young. By Professors W. E. Ayrton, F.R.S., and John Perry, F.R.S.	368
Index	375

[At end of Volume.]

Proceedings at the Meetings, and Report of the President and
Council for the Session 1885-86.

PLATES.

- I. Illustrative of Prof. Fleming's Paper on the Use of Daniell's Cell as a Standard of Electromotive Force.
- II. Illustrative of Mr. Buchanan's Paper on the Thermoelectric Position of Carbon.
- III. & IV. Illustrative of Prof. Fleming's Paper on Problems on the Distribution of Electric Currents in Networks of Conductors treated by the Method of Maxwell.
- V. Illustrative of Mr. Boys's Paper on a Machine for Solving Equations.



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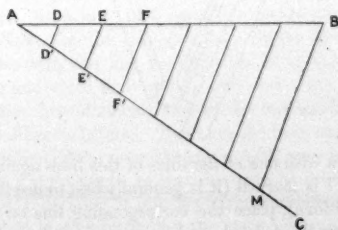
FEBRUARY 1885.

I. *The Uses of a Line-Divider.*

By Miss SARAH MARKS.*

THE ordinary method of dividing a given straight line AB into any number of equal parts (fig. 1), without using a special instrument, is to draw a line AC at any angle to it, to

Fig. 1.



cut off from AC the given number of equal parts AD', D'E', &c., to join M, the last of the points so taken, to B, and from D', E', &c. to draw lines parallel to MB, meeting AB in D, E, &c. These points D, E, &c. divide AB into the required number

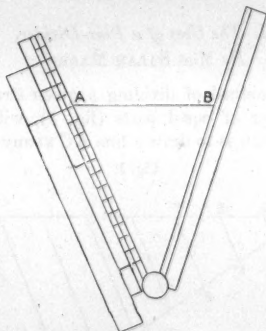
* Read February 14, 1885

of equal parts. This is exactly the method adopted in using the Line-divider.

It consists of a hinged rule with a firm joint, having the left-hand limb fitted to slide in an undercut groove upon the plain rule. Both limbs are bevelled on their inner edges, and the left one is divided both on the bevel and on the top into eighths, quarters, half-inches, and inches, which are consecutively numbered, beginning at the hinged end, so that any set may be used. The plain rule has two needle-points on its underside to prevent it from slipping when placed in any position, and on the top has a single line drawn perpendicular to its longer edges.

To divide the line AB into any number of equal parts, say seven (fig. 2): slide the plain rule along the divided limb till its

Fig. 2.



line coincides with one of the lines of this limb against which the number 7 is marked (it is generally best to use the largest possible divisions), place the corresponding line on the bevel of the limb on the point A, and open or close the rule till the bevel of the undivided limb is on the point B. Now press the plain rule down so that the needle-points enter the paper, and keep it in position. Slide the hinged rule up till the line numbered 6 on the line of reference is coincident with the line on the plain rule, and mark the point where the bevel of the undivided limb meets the line AB. Continue to move the hinged rule up one division at a time till the whole line is

divided. It is evident that the ordinary method has been used; for the divided limb forms a line at some angle to AB, and is divided into seven equal parts, and against the undivided limb I can draw parallel lines through the points of section of the divided limb cutting AB in points equidistant from one another.

This is the first method of using the instrument, and is useful in many ways:—to artists, for squaring out their canvases when they wish to enlarge or diminish their drawings, although the second method would perhaps be best for this if a very large divider were used; to decorators, who frequently find it necessary to divide lines into given numbers of equal parts in order to get patterns into a certain space; it would also be useful for finding the divisions on the scales of thermometers, as the distance between the freezing- and boiling-points is always an unknown length.

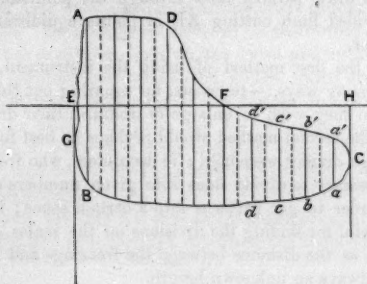
The second method of using the instrument is this:—Let ABCD be an area to be divided into any number of parts, say five, by equidistant lines parallel to AD or BC. Produce AD both ways: place the bevelled edge of the undivided limb along BC, open or close the rule till the end of one of the lines marked 5 on the bevel of the divided limb is on the line AD. Slide the plain rule up, taking care that it is held high enough for its points not to tear the paper, till its line coincides with the 5-line, and press the points firmly into the paper. Now slide the hinged rule up till the division marked 4 coincides with the line on the plain rule, and draw a line along the undivided limb cutting AB and CD. Bring each of the other three divisions against the line on the plain rule, and draw lines as before. These are the lines required.

It is evident that, since the angle ABC may be any angle, it may be a right angle, so that lines can be drawn perpendicular to AB, dividing it into any given number of equal parts. This method is used in finding the mean pressure in gas- and steam-engines by means of an indicator-diagram.

Let ABCD (fig. 3) be an indicator-diagram, EF the atmospheric line; GE, HC, tangents at the extremities of the diagram perpendicular to EF. To find the length of the line which represents the mean pressure of the engine, divide the area EHCG into ten parts by lines parallel to HC and EB, and equidistant from one another; bisect each of the areas

thus obtained by lines parallel to the last. Let the points where these last lines meet the diagram be $a a'$, $b b'$, &c.

Fig. 3.



Add together the lengths $a a'$, $b b'$, &c., and divide by 10; the result will give the line representing the mean pressure of the engine. This process of dividing the diagram, first by ten equidistant lines and then bisecting the spaces between these lines, is generally a very troublesome one, effected by means of a system of parallel rulers and a T-square. In the figure the dotted lines are those not required when the line-divider is used.

To divide the diagram by means of the rule, place the rule as if for dividing the area by 20 equidistant lines parallel to HC. Draw the first of these lines, and then every alternate one, and the diagram is divided as required.

This method of dividing areas is also convenient for drawing treads and rises of stairs, joists, roof-timbers.

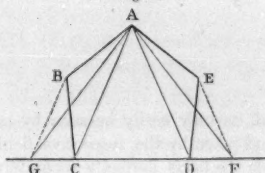
In the third method the divider is used simply as a parallel ruler. As such it has the advantage of having a wide range, and of being firmly fixed at the same time. In graphical statics, where it is often necessary to draw a line parallel to another at a considerable distance from it, set squares have often to be placed and replaced five or six times before the necessary line can be drawn. This instrument will at once draw the line. The range as a parallel ruler may be still further increased by having the plain rule considerably longer than the hinged one.

The special advantages of this instrument as a parallel ruler may best be seen by drawing what surveyors call the "give and take line."

When two fields are divided by a curved boundary, it is often required to find a straight boundary which will divide the fields in the same proportions as before. This straight boundary is what is called the "give and take" line.

In order to understand the method of finding it, it will be necessary to recall the method of finding a rectangle which shall be equal to a given polygon.

Fig. 4.



Let $ABCDE$ (fig. 4) be the given polygon: produce CD both ways, and join AD and AC .

Draw $EF \parallel$ to AD , and $BG \parallel$ to AC , and join AG , AF . Then $\triangle AFG$ is equal in area to the polygon $ABCDE$. For $\therefore \triangle$'s AED and AED are on the same base AD and between the same $\parallel$'s AD , EF ,

$\therefore$ they are equal.

Similarly $\triangle AGC = \triangle ABC$.

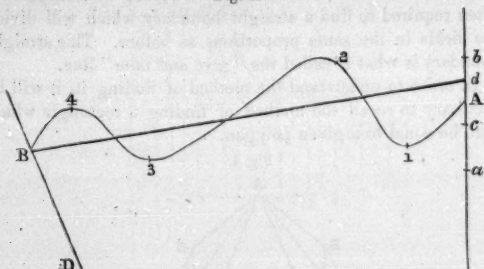
Having this triangle, it is easy to draw a rectangle equal to it.

Now let $A1234B$ (fig. 5) be the curved boundary of a field of which AC , BD are fixed boundaries; it is required to find a straight boundary to replace the curved one through A and B .

Place the divided limb roughly parallel to AC , with its bevelled edge nearest to AC , and at a distance about equal to the breadth of either limb from it, and fix the points firmly in the paper. Now move the undivided limb till it cuts AC at A and 12 at 2 ; slide it along till it is on the point 1 , and mark the point a where it meets AC . Move it till it is against a and 3 , and slide it along till it is against 2 , marking the point b where it meets AC produced, if necessary.

Move it till it is on *b* and 4, slide it till it is on 3, and mark *c*, the point where it meets AC. Move it on to *c* B, slide back to 4, and mark point *d* where it meets AC; join *d* B. *d* B is

Fig. 5.



the line required, as may easily be seen by comparing the portions given and taken by the respective fields. This line would only divide the fields perfectly accurately as before, if A 1 2, 2 3 4, &c., were rectilinear angles; but it is the nearest approximation possible, and the one always used.

The divider would also be useful on board ships for drawing lines on charts parallel to other lines at some distance. It would be much better for this purpose than an ordinary parallel ruler on account of its very large range and capability of being fixed.

Of course the line-divider can be used in all cases in which parallel rulers are ordinarily used.

In summing up, the advantages of the line-divider appear to be these:—

1. Lines of division can be drawn at any angle to the line to be divided.
2. Areas can be divided by equidistant lines parallel to any given line.
3. Since the divisions on the instrument are all equal, they may conveniently be made inches and parts of an inch; and
- 4, consequently the number of divisions may be made as great as we like without materially increasing the cost.
5. Lastly, the cost of the instrument is small, and it is made by Messrs. Stanley in three sizes—6, 12, and 24 inches in length.

II. *On an Integrating Anemometer.*

By WALTER BAILY, M.A.*

To this anemometer, which had been already described (Phys. Soc. Proc. vol. vi. p. 115), mechanical counters similar to those used for bicycles had been added, as it has been found that the electric counters with which the instrument was originally made were liable to let the batteries run down when the wind dropped, and were therefore only suited for observations of short duration.

III. *On the Quadrant-Electrometer.*

By J. HOPKINSON, D.Sc., F.R.S.†

IN Professor Clerk Maxwell's 'Electricity' (vol. i. p. 273) it is proved that the deflection of the needle of a quadrant-electrometer varies as $(A-B)\left(C - \frac{A+B}{2}\right)$, where C is the potential of the needle, and A and B of the two pairs of quadrants. Desiring to ascertain the value of the standard charge of my instrument, I endeavoured to do so by the aid of this formula, and also by a more direct method. The results were quite discordant. Setting aside the special reasoning by which the formula is attained, we should confidently expect that the sensibility of a quadrant-electrometer would increase continuously as the charge of the jar is increased, until at last a disruptive discharge occurs. In my instrument this is not the fact. As the charge was steadily increased by means of the replenisher, the deflection of the needle due to three Daniell's elements at first increased, then attained a maximum, and with further increase of charge actually diminished. On turning the replenisher in the inverse direction the sensibility at first increased, attained the maximum previously observed, and only on further reduction of charge diminished.

Before giving the experimental results, it may be worth while to briefly examine the theory of the quadrant-electrometer. Let A, B, C, D be the potentials of the quadrants, the needle,

* Read February 14, 1885.

† Read March 14, 1885.

and the inductor which is used for measuring high potentials (see Reprint of Sir W. Thomson's papers, p. 278). Let Q_1, Q_2, Q_3, Q_4 be the quantities of electricity on these bodies respectively, and θ the angle of deflection of the needle, measured in terms of divisions of the scale, on which the image of the lamp-flame is projected. We have the equations

$$\left. \begin{aligned} Q_1 &= q_{11}A - q_{12}B - q_{13}C - q_{14}D, \\ Q_2 &= -q_{12}A + q_{22}B - q_{23}C - q_{24}D, \\ Q_3 &= -q_{13}A - q_{23}B + q_{33}C - q_{34}D, \\ Q_4 &= -q_{14}A - q_{24}B - q_{34}C + q_{44}D. \end{aligned} \right\} \quad (1)$$

q_{11} &c. are the coefficients of capacity and induction. They are independent of A, B, C, D , and are functions of θ only. As above written, they are all positive. Let the energy of electrification be W —

$$\left. \begin{aligned} 2W &= q_{11}A^2 + q_{22}B^2 + q_{33}C^2 + q_{44}D^2 \\ &\quad - 2q_{12}AB - 2q_{13}AC - 2q_{14}AD \\ &\quad - 2q_{23}BC - 2q_{24}BD \\ &\quad - 2q_{34}CD \end{aligned} \right\} \quad (2)$$

Equations (1) and (2) are perfectly general, true whatever be the form of the four bodies.

If the four quadrants completely surround the needle,

$$\left. \begin{aligned} q_{44} &= 0, \\ q_{14}, q_{24}, \text{ and } q_{34} &\text{ are independent of } \theta, \\ q_{23} &= q_{13} + q_{12} \end{aligned} \right\} \quad (3)$$

Now when the electrometer is properly adjusted, the needle will not be deflected when $A=B$, whatever C and A may be.

Hence $A-B$ is a factor of $\frac{dW}{d\theta}$, and we have

$$\left. \begin{aligned} \frac{dq_{11}}{d\theta} + \frac{dq_{22}}{d\theta} &= 2 \frac{dq_{12}}{d\theta}, \\ \frac{dq_{23}}{d\theta} &= 0; \end{aligned} \right\}$$

whence

$$2 \frac{dW}{d\theta} = (A-B) \left(A \frac{dq_{11}}{d\theta} - B \frac{dq_{22}}{d\theta} - C \frac{dq_{12}}{d\theta} \right).$$

This should be true of any electrometer having the above adjustment correctly made.

But by suitably forming the three bodies A, B, C, further relations between the coefficients may be obtained. The condition of symmetry would give us $\frac{dq_{11}}{d\theta} = -\frac{dq_{22}}{d\theta}$; but it is not necessary to assume symmetry. If the circumferential termination of the needle be a circle centre in the axis of suspension (at least near the division of the quadrants), if the needle turn in its own plane, if the quadrants are each approximately a surface of revolution about the axis, and if the radial terminations of the needle be not within the electrical influence of the quadrants within which they are not, conditions closely satisfied in Sir W. Thomson's electrometer,

$$\frac{dq_{11}}{d\theta} = \frac{1}{2} \frac{dq_{13}}{d\theta},$$

$$\frac{dq_{22}}{d\theta} = -\frac{1}{2} \frac{dq_{13}}{d\theta}.$$

If θ be small, we obtain

$$2 \frac{dW}{d\theta} = \alpha(A-B) \left(C - \frac{A+B}{2} \right),$$

the formula in Maxwell.

Returning now to our original equation, we have

$$\begin{aligned} 2W = & q_{11}A^2 + q_{22}B^2 + q_{33}C^2 + q_{44}D^2 \\ & - 2q_{12}AB - q_{33}AC - 2q_{14}AD \\ & - q_{33}BC - 2q_{24}BD \\ & + \alpha\theta(A-B) \left(C - \frac{A+B}{2} \right); \end{aligned}$$

involving in all eight constants, q_{11} &c. being now regarded as representing the values of the coefficients in the zero position.

$$\left. \begin{aligned} Q_1 = & q_{11}A - q_{12}B - \frac{1}{2}q_{33}C - q_{14}D + \alpha\theta(C-A), \\ Q_2 = & -q_{12}A + q_{22}B - \frac{1}{2}q_{33}C - q_{24}D - \alpha\theta(C-B), \\ Q_3 = & -\frac{1}{2}q_{33}A - \frac{1}{2}q_{33}B + q_{33}C + \alpha\theta(A-B), \\ Q_4 = & -q_{14}A - q_{24}B + q_{44}D. \end{aligned} \right\}$$

We may now discuss a variety of important particular cases.

(a) B is put to earth; A then is connected to a condenser, capacity a , charged to potential V: we want to know V from

the reading of the electrometer. Here

$$aV - \frac{1}{2}q_{33}C = (a + q_{11})A - \frac{1}{2}q_{33}C + \alpha\theta(C - A),$$

$$V = A + \frac{q_{11}}{a}A + \frac{\alpha\theta(C - A)}{a}.$$

Neglecting A compared with C , and assuming

$$\theta = \lambda(A - B)\left(C - \frac{A + B}{2}\right),$$

we have

$$V = A \left\{ 1 + \frac{q_{11}}{a} + \frac{\lambda\alpha C^2}{a} \right\}.$$

The apparent capacity of A increases with C .

(b) B is again zero. A is connected to a source, but is disconnected and insulated when the deflection of the needle is θ' ; the final deflection is θ : required the potential V of the source.

$$\begin{aligned} q_{11}V - \frac{1}{2}q_{33}C + \alpha\theta'(C - V) \\ = q_{11}A - \frac{1}{2}q_{33}C + \alpha\theta(C - A), \end{aligned}$$

$$\begin{aligned} V = A + \frac{\alpha(\theta - \theta')C}{q_{11}} \\ = A \left\{ 1 + \frac{\alpha\lambda\left(1 - \frac{\theta'}{\theta}\right)C^2}{q_{11}} \right\}. \end{aligned}$$

We may now consider the methods of varying the sensibility of the instrument (see Reprint of Sir W. Thomson's papers, p. 280). The methods dealt with are those of Sir W. Thomson, somewhat generalized.

(c) The quadrant B is connected with an insulated condenser, capacity b , whilst A is connected to a source of electricity:—

$$0 = -q_{12}A + (b + q_{22})B - \alpha\theta C,$$

$$\theta = \lambda(A - B)C;$$

therefore

$$0 = -q_{12}A + (b + q_{22})\left(A - \frac{\theta}{\lambda C}\right) - \alpha\theta C;$$

so

$$A = \theta \frac{\frac{b + q_{22}}{\lambda C} + \alpha C}{(b + q_{22}) - q_{12}}.$$

If $b = 0$, we have the first reduced sensibility given by Sir W. Thomson.

(d) All methods of using the inductor may be treated under one general form. Let the quadrants A and B be connected with insulated condensers, capacities a and b ; then connect the inductor to a source, potential V ;

$$\left. \begin{aligned} 0 &= (q_{11} + a)A - q_{12}B - q_{14}V + a\theta C, \\ 0 &= -q_{12}A + (q_{22} + b)B - q_{24}V - a\theta C, \\ 0 &= \lambda(A - B)C; \end{aligned} \right\}$$

$$\begin{aligned} & (+q_{11}q_{22} - q_{12}^2 + aq_{22} + bq_{11} + ab)(A - B) \\ & + \{q_{14}(q_{12} - q_{22} - b) - q_{24}(q_{12} - q_{11} - a)\} V \\ & + \{-2q_{12} + q_{22} + q_{11} + b + a\} a\theta C = 0; \end{aligned}$$

whence we have an expression for V proportional to θ . By a proper choice of a and b , we can make the sensibility as low as we please.

Now the whole of these formulæ rest on the same reasoning as the equation

$$\theta = \lambda(A - B) \left(C - \frac{A + B}{2} \right).$$

I have mentioned that, in my instrument at least, this equation quite fails to represent the facts when C is considerable. It becomes a matter of interest to ascertain when the formula begins to err to a sensible extent. If a constant battery of a large number of elements were available, this would be soon accomplished. I have at present set up only 18 Daniells. I have therefore been content to use the electrometer to ascertain its own charge by the aid of the inductor, using the 18 Daniells as a standard potential. As the charges range as high as 2600 Daniell's elements, the higher numbers can only be regarded as very rough approximations; sufficiently near, however, to indicate the sort of result which would be obtained if more precise methods were used. The first column in the following Table gives the ascertained or estimated charge of the jar of my electrometer in Daniell's elements; the second the deflection in scale-divisions caused by three elements; the third, the coefficient λ , deduced by the formula $\theta = \lambda AC$: this coefficient ought theoretically to be constant.

I.	II.	III.
72	75	0.35
112	118	0.35
136	140	0.35
178	190	0.35
238	239	0.34
303	288	0.32
383	336	0.30
512	391	0.26
616	409	0.22
813	432	0.18
1080	424	0.13
1312	402	0.10
1728	360	0.07
2124	320	0.05
2634	296	0.037
1704	353	0.07
1436	394	0.09
1284	412	0.11
876	436	0.17
684	427	0.21

By connecting the jar and one quadrant to 18 elements and the other quadrant to earth, I obtained 0.356 as the value of λ , making use of the complete equation

$$\theta = \lambda(A - B) \left(C - \frac{A + B}{2} \right).$$

It will be seen that this equation may be trusted until C is over 200 Daniell's elements potential, but that when C exceeds 250 a quite different law rules.

The foregoing was read before the Physical Society a few years ago, but I stopped its publication after the type was set up, because I was not satisfied that my appliances for experiment were satisfactory, or that I could give any satisfactory explanation of the anomaly.

The electrometer had been many times adjusted for various purposes before further experiments were made, so that those which I shall now describe cannot be directly compared with what goes before. The old experiment was first repeated, and the existence of a maximum sensibility again found. On examination, it was found that the needle hung a little low so that it was nearer to the part of the quadrant below it than to that above. It is easy to see that this would produce the anomalous result observed, though there is reason for thinking it is not the sole cause. The effect of the needle being low

is that it will be on the whole attracted downwards; and so the apparent weight hanging on the fibre-suspension and the consequent tension of the fibres will be increased. The increase of the tension will be as the square of the potential C ; and hence the formula for the deflection will be modified to

$$\theta = \frac{\lambda}{1 + kC^2} (A - B) \left(C - \frac{A + B}{2} \right),$$

where k is a constant depending upon the extent to which the position of the needle deviates from its true position of midway between the upper and lower parts of the quadrants. By a proper choice of k , the results I previously obtained are found to agree well with this formula.

The electrometer was next adjusted in the following way:—The needle was raised by taking up the fibres of the suspension and adjusting them to equal tension in the usual way, and the proportionality of sensibility to charge was tested, the charge being now determined in arbitrary units by discharging the jar of the instrument through a ballistic galvanometer. The operation was repeated until the sensibility, so far as this method of testing goes, was proportional to the charge of the jar over a very long range. It was then found that the needle was slightly above the median position within the quadrants. Increased tension of the fibres from electrical attraction does not therefore account for the whole of the facts, although it does play the principal part. The sensibility of the instrument being now at least approximately proportional to the charge of the jar, I proceeded to determine accurately the potential of the jar when charged to the standard as indicated by the idiostatic gauge.

In what follows the quadrants, one of which is under the induction-plate, are denoted by B , the others by A . The quadrants B are connected to the case, A are insulated. The jar is connected to the induction-plate, and the reading on the scale noted; the connection is broken, and the induction-plate is connected to the case, and the reading on the scale again noted; the difference is the deflection due to the charge in the jar. It is necessary to read the scale for zero-charge on induction-plate last, because the charging of the induction-plate slightly diminishes the charge of the jar, and considerably displaces the zero-reading by giving an induc-

tive charge to the quadrant A. It is also necessary to begin with the charge of the jar minutely too high, so that after separating the induction-plate from the interior of the jar, the latter shall have exactly the correct charge as indicated by the gauge. The deflection thus obtained was precisely $298\frac{1}{2}$, repeated in many experiments. The *double* deflection given by seventy Daniell cells was 43·6 scale-divisions. By comparison with two Clark's cells, the value of which I know, the potential of the seventy Daniells was found to be 74·2 volts; hence the potential of the jar is 1016 volts, when charged to the potential indicated by the gauge.

The constant λ of the instrument was next determined by the formula

$$\theta = \lambda(A - B) \left(C - \frac{A + B}{2} \right).$$

Four modes of connecting are available for this:—

$$A = C = 74\cdot2 \text{ volts, } B = 0;$$

$$B = C = 74\cdot2 \text{ volts, } A = 0;$$

$$A = C = 0, B = 74\cdot2 \text{ volts};$$

$$B = C = 0, A = 74\cdot2 \text{ volts.}$$

In each case the deflection was 253·5 if the charge on the needle was positive in relation to the quadrant with which it was not connected; and was 247 when the needle was negative. This at first appeared anomalous; but the explanation is very simple. The needle is aluminium, the quadrants are either brass or brass-gilded, I am not sure which. There is therefore a contact-difference of potential between the needle and the quadrants; call it x . Thus, instead of $\theta = \frac{A^2}{2}$, we have

$$\theta = +\lambda A \left(\frac{A}{2} + x \right)$$

and

$$\theta = \lambda(-A) \left(-\frac{A}{2} + x \right);$$

this gives

$$x = \frac{6\cdot5}{2\lambda A} = 0\cdot482 \text{ volt.}$$

The result was verified by using fourteen cells instead of seventy: the deflections were 10·0 and 8·8, which gives the same value to x . It is worth noting that the same cause affects the idiostatic gauge in the same way. Let the jar be charged till the gauge comes to the mark. Call P the differ-

ence of potential between the aluminium lever of the idiostatic gauge and the brass disk below which attracts it. The difference of potential between the brass of the case and the brass work of the interior is $P+x$, and between the case and the aluminium needle within the quadrants it is $P+2x$. If, however, the charge is negative, the difference is $-P+2x$. Hence the sensibility will be different from two causes, according as the jar is charged positively or negatively, till the idiostatic gauge is at its standard. For determining the constant λ we must take the mean of the two results 253.5 and 247, that is 250.25. Comparing this with the actual standard charge of the jar, and the double deflection given by one volt 172.4, when charged to the standard, we see that the irregularity has not been wholly eliminated. It appeared desirable to determine the sensibility of the instrument for a lower known charge. The charge was determined exactly as described above, and was found to be 609 volts; whilst 1 volt gave 107.1 scale-divisions double deflection; whence in the equation

$$\theta = \frac{\lambda(A-B)C}{1+kC^2},$$

we have, if

$$\lambda = 0.1816, \quad k = 7 \times 10^{-8},$$

the following as the calculated and observed deflections:—

Calculated	250.0,	107.7,	172.4
Observed	250.2,	107.1,	172.4,

which is well within errors of observation.

This deviation from proportionality of sensibility did not appear to be worth correcting, as I was not sure that other small irregularities might not be introduced by raising the needle above the middle position within the quadrants. It appears probable that the small deviation still remaining does not arise from the attraction of the quadrants on the needle increasing the tension of the suspension, but from some cause of a quite different nature, for if it were so caused the capacity-equations would be

$$\theta_1 = q_{11}A - q_{12}B - q_{33}C - q_{14}D + a\theta C, \text{ \&c.,}$$

where

$$\theta = \frac{\lambda}{1+kC^2}(A-B) \left(C - \frac{A+B}{2} \right).$$

Now the experiments I have tried for determining q_{11} , q_{12} , &c., are not in accord within the limits of errors of observation, using these equations of capacity; but they are in better accord if, in lieu of the term $\alpha\theta C$, we write $\frac{\alpha\theta C}{1+kC^2}$. I have no explanation of this to offer; but in what follows it is assumed that the equations expressing the facts are

$$\left. \begin{aligned} \theta &= \mu(A-B), \text{ where } \mu = \frac{\lambda C}{1+kC^2}, \\ Q_1 &= q_{11}A - q_{12}B - q_{14}D + \beta\mu\theta, \\ Q_2 &= -q_{12}A + q_{22}B - q_{24}D - \beta\mu\theta, \\ Q_4 &= -q_{14}A - q_{24}B + q_{44}D. \end{aligned} \right\}$$

We are now in a position to determine the various coefficients of capacity: in doing so it is necessary to distinguish the values of q_{11} and q_{22} when the posts by which contact with the quadrants is made are down and in contact with the quadrants, and when they are raised up out of contact; the former are denoted by $q_{11}+a$ and $q_{22}+a$, the latter by q_{11} and q_{22} , the capacity of the binding-posts being a . As a convenient temporary unit of capacity the value of $\beta\mu^2$, when the jar has the standard charge, is taken. The first set of experiments was to determine the deflections caused by known potentials with varied charge of jar, one or other of the quadrants being insulated. Three potentials of the jar were used—that of the standard indicated by the idiostatic gauge and two lower. The values of μ are denoted by μ_3 , μ_2 , μ_1 . It was found by connecting the two quadrants to standard cells that

$$\mu_3 : \mu_2 : \mu_1 = 1 : 0.805 : 0.585;$$

and hence

$$\beta\mu_3^2 = 1, \beta\mu_2^2 = 0.648, \beta\mu_1^2 = 0.342.$$

Suppose quadrant A be insulated, and potential B be applied to quadrant B; then we have, if θ be deflection which potential B would cause with standard charge, if quadrant A were connected to the case, and ϕ the observed deflection,

$$0 = q_{11}A - q_{12}B + \beta\mu\phi;$$

$$\phi = \mu(A-B);$$

$$\theta = -\mu_3 B;$$

whence

$$\phi = \theta \cdot \frac{\mu}{\mu_3} \cdot \frac{q_{11} - q_{12}}{q_{11} + \beta\mu^2}.$$

In the calculated values of ϕ given below,

$$\begin{aligned} q_{11} &= 0.502, & q_{22} &= 0.543 \\ q_{12} &= 0.293, & a &= 0.200 \text{ for B,} \\ & & &= 0.193 \text{ for A.} \end{aligned}$$

A closer approximation to observation is obtained by assuming the two contact-posts to be of slightly different capacities; the difference given above is no more than might be expected to exist.

The jar being charged to standard potential, B was insulated, and its post raised, and A was connected to 10 Daniells, for which $\theta = 1808$:—

$$\begin{aligned} \text{Deflection observed} &= 293.2, \\ \text{,, calculated} &= 293.0. \end{aligned}$$

The post of B was lowered to contact :—

$$\begin{aligned} \text{Deflection observed} &= 467.0, \\ \text{,, calculated} &= 466.8. \end{aligned}$$

A was now insulated and post raised, B was connected to the same battery :—

$$\begin{aligned} \text{Deflection observed} &= 251.0, \\ \text{,, calculated} &= 251.6. \end{aligned}$$

The post of A was lowered to contact :—

$$\begin{aligned} \text{Deflection observed} &= 429.0, \\ \text{,, calculated} &= 428.8. \end{aligned}$$

The jar was now charged to a lower potential, for which $\mu = \mu_2$ with B insulated and post raised, and A connected to 30 Daniells, for which $\theta = 5468$:—

$$\begin{aligned} \text{Deflection observed} &= 925.0, \\ \text{,, calculated} &= 924.0. \end{aligned}$$

The post of B was lowered to contact, and A connected to 10 Daniells, for which $\theta = 1808$:—

$$\begin{aligned} \text{Deflection observed} &= 470.5, \\ \text{,, calculated} &= 470.85. \end{aligned}$$

A was now insulated and post raised, B was connected to

a battery of 30 Daniells, for which $\theta=5468$:—

Deflection observed = 798.0,

„ calculated = 800.0.

The post of A was lowered to contact, and B was connected to 10 Daniells ; $\theta=1808$:—

Deflection observed = 437.0,

„ calculated = 435.7.

The jar was then charged to a still lower potential, for which $\mu=\mu_1$, with B insulated and post raised, and A connected to 30 Daniells, for which $\theta=5468$:—

Deflection observed = 901.0,

„ calculated = 903.6.

The post of B was lowered to contact and A connected to 10 Daniells ; $\theta=1808$:—

Deflection observed = 437.0,

„ calculated = 438.7.

A was now insulated and post raised, and B was connected to 30 Daniells ; $\theta=5468$:—

Deflection observed = 785,

„ calculated = 792.

The post of A was lowered to contact and B connected to 10 Daniells ; $\theta=1808$:—

Deflection observed = 408,

„ calculated = 410.

The next experiment was similar, excepting only that the insulated quadrant B was connected to a condenser ; this condenser consisted merely of a brass tube insulated within a larger tube—its capacity is about 0.00009 microfarad. The jar was at its standard charge. Calling the capacity of the condenser b , in terms of our temporary unit, we have, as before,

$$\phi = \theta \frac{b + a + q_{22} - q_{12}}{b + a + q_{22} + 1} [\beta \mu_3^2 = 1].$$

When $\theta=1259$, ϕ was observed to be 927, whence $b=3.159$.

We are now in a position to obtain independent verification of the values already obtained for the constants. Suppose A be connected to the case, that condenser b is charged from

a battery of known potential, such that it would give deflection θ if connected to B, and the charged condenser is then connected to B. Suppose ψ be the deflection before connection is made, ϕ after. Then

$$b(\theta - \phi) = \{q_{22} + a + 1\}(\phi - \psi).$$

When $\theta = 1439$ and $\psi = 0$, it was found that $\phi = 915$. The value of ϕ , calculated from the values of the constants already obtained, is 928.

When $\theta = 1439$ and $\psi = -676$, it was found that $\phi = +676$; the calculated value is 688.

A further experiment of verification, involving only the capacity of the quadrant, is the following. The quadrant A being connected to the case, B was charged by contact instantaneously made and broken with a battery of known potential, and the resulting deflection was noted. The instantaneous contact being made by hand, no very great accuracy could be expected. Let ψ and ϕ be the readings on the scale before and after the instantaneous contact; then

$$\frac{\theta - \phi}{\phi - \psi} = \frac{1}{q_{22} + a} = 1.345.$$

The following results were obtained:—

θ .	ψ .	ϕ observed.	ϕ calculated.
1796	0	763	765
1796	-493	493	482

We next determine the coefficients q_{14} and q_{24} of induction of the induction-plate on the quadrants. This is easily done from the deflections obtained with the induction-plate, one or both pairs of quadrants being insulated. First, suppose one pair, say B, are insulated whilst A is connected to the case:—

$$0 = +q_{22} B - q_{24} D - \beta\mu\phi,$$

$$\phi = -\mu B,$$

$$\theta = \mu_2 D;$$

whence

$$\phi = \theta \frac{\mu}{\mu_2} \frac{q_{24}}{q_{22} + \beta\mu^2},$$

ϕ being the deflection actually observed, and θ that which
2 c

the battery used would give if connected direct to the quadrants, the needle having the standard charge. When θ was 12,800 and $\mu = \mu_3$, ϕ was 418, whence $q_{24} = 0.0504$.

In the same way A being insulated but B connected to the case, ϕ was found to be 43.6, whence $q_{14} = 0.00508$.

Again, when both quadrants are insulated we have

$$0 = q_{11}A - q_{12}B - q_{14}D + \beta\mu\phi,$$

$$0 = -q_{12}A + q_{22}B - q_{24}D - \beta\mu\phi,$$

$$\phi = \mu(A - B),$$

$$\theta = \mu_3 D.$$

From the first two equations,

$$(q_{11}q_{22} - q_{12}^2)(A - B) - \{(q_{22} - q_{12})q_{14} - (q_{11} - q_{12})q_{24}\}D$$

$$+ (q_{22} + q_{11} - 2q_{12})\beta\mu\phi = 0;$$

whence

$$\phi = \frac{\theta\mu}{\mu_3} \frac{(q_{22} - q_{12})q_{14} - (q_{11} - q_{12})q_{24}}{(q_{11}q_{22} - q_{12}^2) + (q_{22} + q_{11} - 2q_{12})\beta\mu^2}.$$

In the case when $\mu = \mu_3$, substituting the values already determined, we have

$$\phi = \theta \times 0.0142;$$

it was observed with $\theta = 12,800$ that $\phi = 183$; the calculated value would be 182.

With a lower charge on the jar, viz. when $\mu = \mu_3 \times 0.805$, with B insulated, A connected to the case, and $\theta = 12,800$, it was found that $\phi = 437.5$; the calculated value is 441.

The capacity q_{44} of the induction-plate is of no use; its value, however, is about 0.004, in the same unit as has been so far used.

The capacity q_{33} of the needle and the coefficient of induction of the needle on either quadrant $\frac{1}{2}q_{33}$ are also of no use, but the method by which they may be obtained is worth noting. Let quadrants A be connected to the case, and let B be insulated, diminish the charge of the jar slightly by the replenisher, and suppose the consequent deflection be ϕ . Let μ and μ' be the values of μ before and after the diminution of charge, as ascertained by applying a known potential-

difference between the two pairs of quadrants; we have

$$-q_{33}C = +q_{22}B - q_{33}C' - \beta\mu'\phi,$$

where

$$\phi = -\mu'B,$$

$$q_{33}(C - C') = (q_{22} + \beta\mu'^2)B,$$

which determine q_{33} , since C and C' are known from μ and μ' .

Of course the values of the constants of an electrometer are of no value for any instrument except that for which they are determined in the state of adjustment at the time. For any particular use of the instrument it is best to determine exactly that combination of constants which will be needed. Nor is there anything new in principle in the discussion or experiments here given; they are merely for the most part the application of well-known principles to methods of using the electrometer given by Sir William Thomson himself. The method of determining the capacity of a condenser by charging it and connecting it to an insulated quadrant has been used by Boltzmann. But the invention of the quadrant-electrometer by Sir W. Thomson may be said to have marked an epoch in Electrostatics, and the instrument from time to time finds new uses. It therefore seems well worth while to make known observations made upon it in which the instrument itself has been the only object studied. Some practical conclusions may, however, be drawn from the preceding experiments. Before using the formula

$$\theta = \lambda(A - B) \left(C - \frac{A + B}{2} \right)$$

it is necessary to verify that it is sufficiently nearly true, or to determine its variation from accuracy. Unless it be sufficiently accurate through the range experimented upon, the electrometer cannot be applied by the methods well known for determining alternating potentials and the work done by alternating currents.

My pupil, Mr. Paul Dimier, has very efficiently helped me in the execution of the experiments of verification.

IV. *Notes on the Use of Nicol's Prism.* By JAMES C. M'CONNEL, B.A., Assistant Demonstrator at the Cavendish Laboratory, Cambridge*.

1. *On the Error in the Measurement of a Rotation of the Plane of Polarization caused by the Axis, about which the Nicol turns, not being parallel to the Incident Light.*

SUPPOSE we have a beam of parallel light traversing a Nicol's prism mounted in a graduated circle. Unless we have taken special precautions, we shall find that, when the Nicol is rotated, the plane of polarization of the emergent light turns through an angle somewhat different from that measured by the circle. For instance, if the axis of rotation of the Nicol is inclined 3° to the direction of the incident or emergent light, as large an error as 1° may be made in measuring a rotation of 60° .

It is, however, tolerably well known that, as far as this error is proportional to the first power of the small angle of deviation, it may be eliminated by taking the mean of the readings in the two opposite positions of the Nicol circle separated from one another by nearly two right angles. It is of course only to a first approximation that this proportionality can be considered to hold good. And the main object of the present investigation is to determine, what is the outstanding error we are liable to in assuming this proportionality; or, in other words, with what accuracy we must adjust the Nicol circle, that this first approximation may be sufficient.

The phenomenon in its simplest form may be described thus:—If the emergent ray be parallel to the long edges of the prism it is polarized in a plane perpendicular to the principal plane of the prism, i. e. to the plane containing the optic axis and the normal to the face. If the ray be inclined to this position but still lie in the principal plane, then the new plane of polarization is as nearly parallel to the old one as is consistent with its containing the new ray. This indeed is obvious from symmetry. But now let the ray lean out of the principal plane: there is a marked change in the plane of

* Read February 28, 1885.

polarization. It has been twisted about the ray, and the angle of twist is nearly one third of the angle of deviation. From this explanation it is easy to see in a general way how the error arises, and also how it may be eliminated by taking readings in the two opposite positions of the Nicol.

There is another important point that is apparent on the face of the matter. If the direction of the ray coincide with the axis of rotation, the ray will not move relatively to the prism when the prism is turned, and the rotation measured by the circle will be identical with the actual rotation of the plane of polarization. We shall find, moreover, that to the second order of approximation the final error depends entirely on the angle between these two directions (see equation 8). It is independent of any reasonably small errors in the adjustment of the Nicol in the circle.

In a Nicol's prism used as a polarizer, it is the second half of the prism that determines the plane of polarization of the emergent light. If the optic axis of the spar in the first half be not accurately parallel to the axis in the second half, the effect is not to turn the plane of polarization, but merely to mix a little ordinary light with the emergent polarized light.

The figure represents a portion of the sphere of unit radius.

X is the optic axis. N the normal to the face.

So XN is the principal plane.

P' is the wave-normal of the emergent light.

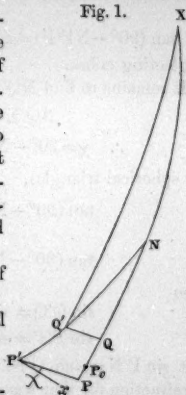
Q' is the wave-normal of the internally incident light.

Draw Q'Q, P'P perpendicular to XN produced. Join XQ'.

Let P'x be the plane of polarization of P' and $PP'x = \chi$. Then χ is a function of P'P and PN.

The exact function would be a very complicated expression, so we shall content ourselves with an approximation. P'P is always small, and P lies near some fixed point, P_0 say, in the

Fig. 1.



plane XN. Let

$$P'P = \alpha, \quad PP_0 = \beta.$$

We shall reject the cubes and higher powers of α and β and limit ourselves to expanding χ as far as the squares. We know that when α is zero, χ is zero; so our expression can only contain the terms α , α^2 , and $\alpha\beta$.

In a Nicol's prism it is the extraordinary ray which emerges, so the plane of polarization of Q' is perpendicular to QX . Let us assume for the present that the plane of polarization of P' makes the same angle with the plane of incidence $P'Q'N$ as does the plane of polarization of Q' . The plane of polarization of Q' makes with $P'Q'N$ an angle $90^\circ - NQ'X$. Hence

$$\chi = 90^\circ - NQ'X - NP'P. \quad (1)$$

But in the triangle $NP'P$, P is a right angle; so we have

$$\tan(90^\circ - NP'P) = \cot NP'P = \frac{\sin P'P}{\tan NP} = \frac{\alpha}{\tan NP}, \quad (2)$$

neglecting cubes.

It remains to find $NQ'X$.

$$NQ'X = XQ'Q - NQ'Q,$$

$$\therefore \chi = 90^\circ - NP'P + NQ'Q - XQ'Q. \quad (3)$$

By spherical triangles,

$$\left. \begin{aligned} \tan(90^\circ - XQ'Q) &= \frac{\sin Q'Q}{\tan XQ}, \\ \tan(90^\circ - NQ'Q) &= \frac{\sin Q'Q}{\tan NQ}. \end{aligned} \right\}$$

Also

$$\left. \begin{aligned} \sin Q'Q &= \sin Q'N \sin N, \\ \sin P'P &= \sin P'N \sin N. \end{aligned} \right\}$$

But $\sin P'N = \mu \sin Q'N$, where μ is the extraordinary index of refraction for that wave, which we may take to be constant.

$$\therefore \sin Q'Q = \frac{1}{\mu} \sin P'P = \frac{\alpha}{\mu} \text{ to our order of approximation.}$$

Hence

$$\left. \begin{aligned} \tan(90^\circ - XQ'Q) &= \frac{\alpha}{\mu} \frac{1}{\tan XQ}, \\ \tan(90^\circ - NQ'Q) &= \frac{\alpha}{\mu} \frac{1}{\tan NQ}. \end{aligned} \right\} \quad (4)$$

Since $\tan XQ$ and $\tan NQ$ occur in small terms, we may reject squares in finding their values.

In NQ let us take a point Q_0 such that $\sin NP_0 = \mu \sin NQ_0$.
Then, since $\sin(NP_0 + \beta) = \mu \sin(NQ_0 + QQ_0)$,

$$\beta \cos NP_0 = \mu QQ_0 \cos NQ_0,$$

and

$$\left. \begin{aligned} \frac{1}{\tan XQ} &= \frac{1}{\tan XQ_0} - \frac{QQ_0}{\sin^2 XQ_0} = \frac{1}{\tan XQ_0} - \frac{\beta \cos NP_0}{\mu \cos NQ_0 \sin^2 XQ_0} \\ \frac{1}{\tan NQ} &= \frac{1}{\tan NQ_0} - \frac{QQ_0}{\sin^2 NQ_0} = \frac{1}{\tan NQ_0} - \frac{\beta \cos NP_0}{\mu \cos NQ_0 \sin^2 NQ_0} \end{aligned} \right\} (5)$$

Now in the expansion of the tangent of a small angle the squares of the angle do not appear. So we have by (3), (2), and (4),

$$\chi = \frac{\alpha}{\tan NP} + \frac{\alpha}{\mu} \frac{1}{\tan XQ} - \frac{\alpha}{\mu} \frac{1}{\tan NQ},$$

while

$$\frac{1}{\tan NP} = \frac{1}{\tan NP_0} - \frac{\beta}{\sin^2 NP_0}$$

So by (5),

$$\begin{aligned} \chi &= \alpha \left\{ \frac{1}{\tan NP_0} + \frac{1}{\mu \tan XQ_0} - \frac{1}{\mu \tan NQ_0} \right\} \\ &+ \alpha \beta \left\{ -\frac{1}{\sin^2 NP_0} - \frac{\cos NP_0}{\mu^2 \cos NQ_0} \left(\frac{1}{\sin^2 XQ_0} - \frac{1}{\sin^2 NQ_0} \right) \right\} (6) \\ &= l\alpha - m\alpha\beta \text{ say.} \end{aligned}$$

In Nicol's prism, as it is usually cut,

$$NX = 41\frac{1}{2}^\circ,$$

$$NP_0 = 22^\circ,$$

$$\mu = 1.54,$$

$$\therefore NQ_0 = 14^\circ.$$

Substituting these values, we obtain

$$l = .32, \quad m = .84;$$

$$\chi = .32\alpha - .84\alpha\beta. \quad \dots \dots (7)$$

In the foregoing we have assumed that there is no rotation of the plane of polarization on refraction out of the spar, or, in other words, that the angle between the plane of polarization and the plane of incidence remains unchanged. There is of course a change, but it is merely due to the disturbing effect of reflection and is very small. If we treat the spar as a homogeneous medium and use Fresnel's formulæ, we find the rotation is about $.005\alpha$ in the case of an ordinary Nicol.

I have calculated from Neumann's theoretical formulæ for refraction at the surface of crystals what the rotation would be in the case of light entering the Nicol and exciting only the extraordinary wave, and find it is less than with a homogeneous medium. In the case of light leaving the Nicol there are two reflected waves, so the formulæ would probably be very complicated. I think we may safely assume that the rotation is less than $\cdot 01\alpha$, and may therefore be neglected. In the case of a flat-ended Nicol the incidence is nearly direct, and the rotation may obviously be neglected.

Owing to the circumstance that this rotation is negligible, the whole of this investigation applies without alteration to the case when the Nicol is used as an analyzer, and the light is consequently incident on the spar. In this case we have to find the position of the plane of polarization that only the ordinary wave may be excited, and the formula for the rotation is the same as in an isotropic medium.

We are now in possession of a convenient formula for expressing the position of the plane of polarization in terms of the direction of the emergent light relative to fixed planes in the Nicol. We shall apply this formula to the discussion of the error in measuring a rotation. This part of the investigation would be very short if we confined ourselves to the first approximation. A number of complications are introduced by the necessity of retaining the squares and products of small quantities.

The figure represents a portion of the sphere of unit radius. Let

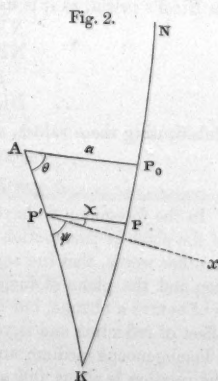
A be the axis of rotation,

NP the principal plane,

P' the emergent wave-normal,

P'x the plane of polarization.

Draw AP_0 and $P'P$ perpendicular to the principal plane. This fixes P_0 , which has hitherto been arbitrary within certain limits. A and P' are fixed in space, while N and P_0 are rotated round A.



For simplicity, suppose the circle-reading is zero when F' lies on $A P_0$. Then $\theta = P_0 A P'$ is the reading at any time.

Produce $A P'$ to meet $N P_0$ in K .

What we want to measure is the rotation of $P'x$ round P' ; so let $x P' K = \psi$. As before, let $P'P = \alpha$, $PP_0 = \beta$, and let $AP_0 = a$, $AP' = r$. Then α , β , a , r are all small. Since the figure $A P_0 P P'$ is small, we have to our order of approximation

$$\begin{aligned} \alpha &= a - r \cos(\psi + \chi), \\ \beta &= r \sin(\psi + \chi). \end{aligned} \quad \}$$

Let us now find the difference between θ and $\psi + \chi = PP'K$. By spherical triangles,

$$\begin{aligned} \cos K &= \sin(\psi + \chi) \cos PP', \\ \cos K &= \sin \theta \cos a; \end{aligned} \quad \}$$

$$\therefore \sin(\psi + \chi) \cos PP' = \sin \theta \cos a,$$

$$\sin \theta = \sin(\psi + \chi) \left(1 - \frac{\alpha^2}{2} + \frac{a^2}{2}\right),$$

$$\sin(\psi + \chi) + (\theta - \psi - \chi) \cos(\psi + \chi) = \sin(\psi + \chi) \left(1 - \frac{\alpha^2}{2} + \frac{a^2}{2}\right);$$

$$\therefore \theta - \psi - \chi = \frac{a^2 - \alpha^2}{2} \tan(\psi + \chi)$$

$$= \frac{a^2 - \alpha^2}{2} \tan \psi, \text{ neglecting cubes.}$$

But

$$a^2 - \alpha^2 = 2ar \cos \psi - r^2 \cos^2 \psi,$$

and

$$\chi = l\alpha - ma\beta$$

$$= l(a - r \cos \psi + \chi) - mr \sin \psi (a - r \cos \psi)$$

$$= l\{a - r \cos \psi + lr \sin \psi (a - r \cos \psi)\} - mr \sin \psi (a - r \cos \psi)$$

$$= la - lr \cos \psi + (l^2 - m)r \sin \psi (a - r \cos \psi);$$

$$\therefore \theta = \psi + la - lr \cos \psi + (l^2 - m + 1)ar \sin \psi$$

$$- (l^2 - m + \frac{1}{2})r^2 \sin \psi \cos \psi.$$

Now let the plane of polarization be turned through 180° and the new reading of the Nicol circle be $180^\circ + \theta_1$. Instead of ψ we must write in the last formula $180 + \psi$. So

$$\theta_1 = \psi + la + lr \cos \psi - (l^2 - m + 1)ar \sin \psi$$

$$- (l^2 - m + \frac{1}{2})r^2 \sin \psi \cos \psi;$$

$$\therefore \frac{\theta + \theta_1}{2} = \psi + la - (l^2 - m + \frac{1}{2})r^2 \sin \psi \cos \psi,$$

$$\frac{\theta_1 - \theta}{2} = lr \cos \psi - (l^2 - m + 1)ar \sin \psi.$$

Inserting numerical values, we have

$$\frac{\theta + \theta_1}{2} - \psi = .32a + .24r^2 \sin \psi \cos \psi, \quad . \quad . \quad (8)$$

$$\theta_1 - \theta = .64r \cos \psi, \text{ approximately.} \quad . \quad . \quad . \quad (9)$$

To render these formulæ intelligible to one who has not read through the investigation, we may remark that we have to deal with one plane fixed in space, viz. the plane containing the axis of rotation of the Nicol circle and the direction of the emergent light.

θ and $180^\circ + \theta_1$ are the readings of the circle in the two opposite positions, and $\theta = 0$ when the principal plane of the Nicol is at right angles to the fixed plane.

ψ is the angle between the plane of polarization and the fixed plane.

a is the angle between the axis of rotation and the principal plane.

r is the angle between the axis of rotation and the emergent light.

The angles are supposed to be expressed in circular measure in these as in all the other formulæ.

The first term on the right-hand side of equation (8) is a constant, and is therefore of no consequence in measuring a change of ψ . The second term is a measure of the outstanding error.

Let us suppose

$$r = 3^\circ, \quad \psi = 30^\circ.$$

Then

$$.24r^2 \sin \psi \cos \psi = 1';$$

but if $\psi = -30^\circ$,

$$\text{the same} = -1'.$$

So in measuring a rotation of 60° we may be subject to an error of $2'$ if the axis of rotation be inclined to the emergent light at an angle of 3° . To make sure that the error shall be less than $1'$, the last-mentioned angle must be kept within 2° .

Equation (9) affords the means of deducing the value of r from the difference of readings in the two opposite positions.

The second term of (8) is due to two main causes. One is that the rotation of the plane of polarization takes place about one axis, while the rotation is measured about another. The effect of this appears in the coefficient $-\frac{1}{2}$. The other part of the error is due to the values of χ in the two opposite positions not exactly neutralizing each other. This appears in the coefficient $m-l^2$. If these two causes had reinforced instead of counteracting one another, the resultant error would have been five times as large.

If the arrangements permit of taking readings when the plane of polarization is inclined at 90° to its two first positions, we can, by taking the mean of all four readings, get rid of the error depending on the squares of small quantities, as is evident from (8). This is what Lord Rayleigh has done in his recent measurements of the electromagnetic rotation in bisulphide of carbon.

The formula (6) applies to almost every mode of cutting a Nicol, except the important case when the ends are cut off square to the length, the case of the so-called "flat-ended Nicol." This case requires the investigation to be modified. The peculiarity consists in N lying very close to P' .

$$\chi = 90^\circ - NP'P + NQ'Q - XQ'Q,$$

as before;

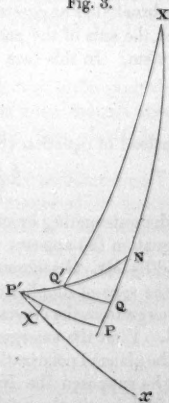
and by (4) and (5),

$$\begin{aligned} & \tan(90^\circ - XQ'Q) \\ &= \frac{\alpha}{\mu} \frac{1}{\tan XQ_0} - \frac{\alpha\beta}{\mu^2 \sin^2 XQ_0}. \end{aligned}$$

If we take N as the point from which to measure β , P_0 and Q_0 will coincide with N .

We have now to find $NP'P - NQ'Q$. The process is precisely similar to one already performed in finding equation (8);

Fig. 3.



and we obtain

$$\begin{aligned} \text{NP'P} - \text{NQ'Q} &= \frac{\text{PP}'^2 - \text{QQ}'^2}{2} \tan \text{NP'P} \\ &= \frac{\mu^2 - 1}{2\mu^2} \alpha\beta; \end{aligned}$$

$$\begin{aligned} \therefore \chi &= \frac{\alpha}{\mu \tan \text{XN}} - \frac{\alpha\beta}{\mu^2 \sin^2 \text{XN}} - \frac{\mu^2 - 1}{2\mu^2} \alpha\beta \\ &= l\alpha - m\alpha\beta, \text{ say.} \end{aligned}$$

Taking $\text{XN} = 63^\circ$, $\mu = 1.52$, we obtain

$$l = .33, \quad m = .83.$$

So the numbers are almost exactly the same as in the case of the ordinary Nicol.

As a test of the accuracy of the above calculations, I made a few observations on a flat-ended Nicol. I found that, turning the Nicol through $30'$, about an axis in the principal plane and at right angles to the incident light, turned the plane of polarization through $11' \pm 1'$.

In a paper on Polarizing Prisms (Phil. Mag. 1883) Mr. Glazebrook has suggested a new form of flat-ended prism in which the axis of the spar is at right angles to the length of the prism. In this case $\text{XN} = 90^\circ$, and $\mu = 1.49$.

$$\therefore l = 0, \quad m = .73,$$

$$m - l^2 - \frac{1}{2} = .23.$$

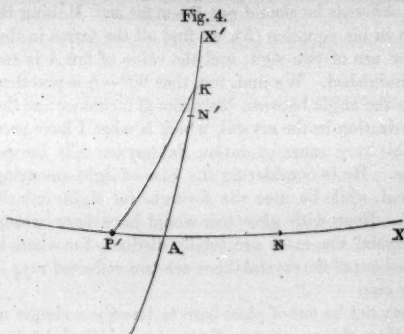
So instead of equation (8) we have

$$\frac{\theta + \theta_1}{2} - \psi = .23r^2 \sin \psi \cos \psi,$$

and the outstanding error is substantially the same as before.

Equation (9) appears to be inconsistent with a result obtained by Mr. Glazebrook (Phil. Mag. Oct. 1880, p. 252). For the sake of simplicity, he supposed the ordinary ray only to emerge from the crystal. He examined two positions of the Nicol. First the emergent ray lay in the principal plane, so that the plane of polarization coincided with the principal plane. Next he supposed the Nicol to be turned through 90° about an axis, lying in the principal plane, but inclined at 5° to the emergent ray, and he found that now the plane of polarization was inclined at $5^\circ 3'$ to the principal plane. *But this does not show that the plane of polarization has been turned through*

$90^\circ \pm 5^\circ 3'$ about the direction of the ray, as is evident on examination of the annexed figure.



P' is, as before, the emergent light; N , N' , X , X' are the two positions of the normal to the face and the axis.

In the first position the plane of polarization is $P'ANX$.

In the second position let it be $P'K$.

Mr. Glazebrook finds $P'KA = 5^\circ 3'$.

The angle, however, that we wish to find, is the angle through which the plane of polarization has been turned about the direction of the ray, viz. $AP'K$, and this, of course, is not equal to $90^\circ - P'KA$.

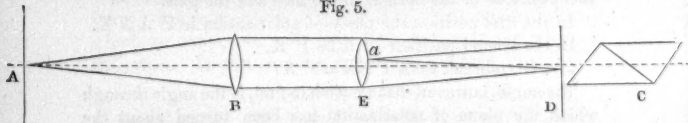
After I had written the greater part of this paper, I found that the ground had been already traversed by Sande Bakhuizen (*Pogg. Ann.* cxlv. p. 259, 1872). His investigation is considerably longer than mine, and he only examines the case of the ordinary Nicol. He obtains results of the same general form; but he falls into at least one serious error; so his numerical values are quite different. Instead of assuming, as I have done, that the rotation of the plane of polarization on refraction out of the crystal is negligible, he uses a complicated formula, giving directly the position of the plane of polarization of the incident ray in which only the extraordinary ray is excited in the crystal. This formula he has apparently deduced from a formula given by Neumann (*Pogg. Ann.* xlii. p. 11). And it is here that the error occurs. He has not

paid sufficient attention to Neumann's explanation of how the quantities in the formula are to be measured. He puts β for ω ; whereas he should put $\beta - \pi$ for ω . Making this correction in his equation (5), we find all the terms in the denominator are of one sign, and the value of $\tan A$ is considerably diminished. We find, too, that $90^\circ - A$ is practically the same as the angle between the plane of incidence and the plane of polarization in the crystal, which is what I have assumed.

In this very same operation Bakhuyzen falls into another mistake. He is considering the case of light emerging from the Nicol, while he uses the formula for light entering the crystal. Even with glass this would have been wrong; but with crystal the cases are totally distinct, for when light is refracted out of the crystal there are two reflected rays instead of only one.

It may not be out of place here to describe a simple method for testing the necessary adjustment, which I lately found convenient. In my arrangement (fig. 5) the source of light

Fig. 5.



was an illuminated slit A, whence the light passed through a lens B, placed so that the slit was at its principal focus. Thus a parallel beam of light fell on the polarizing Nicol C. To the Nicol circle I attached a plane mirror D as nearly as possible at right angles to the axis of rotation. Between B and D, I interposed a lens E at a distance of half its focal length from D. An image of the slit is thus formed at a on the surface of the lens E, or rather on a scrap of paper attached to it. The image a of course remains at the same point on the paper, however the lens E is moved laterally in its own plane. On rotating the Nicol circle I found that a did not move; so I knew that the mirror was at right angles to the axis of rotation. I then adjusted the Nicol circle till the middle point of the image of the slit fell on the optical centre of the lens, which I had previously marked on the paper. If I had

chosen a wrong point as the optical centre, the error would have been at once evident on turning the lens E through 180° in its own plane. This method gave without much difficulty an accuracy of 1° , with a lens E of only 4 inches focal length.

2. *On a new Method of obtaining the Zero-reading of a Nicol Circle.*

In a large class of experiments on polarized light, it is necessary to know the reading of the Nicol circle when the plane of polarization is parallel to the axis of rotation of some part of the apparatus, *e. g.* of the table of a spectrometer. The usual method of obtaining this reading depends on the polarizing power of a glass reflecting surface. It is easy to fix the surface on the spectrometer-table parallel to the axis of rotation. If the angle of incidence be made that of maximum polarization, and the Nicol turned till the reflected light is reduced to a minimum, the plane of polarization is then perpendicular to the plane of incidence. This method is simple, but it is not very sensitive. Even with poor illumination the minimum reflected light is, with glass at any rate, by no means evanescent; and as the Nicol is turned the intensity remains sensibly constant for some distance on either side of the minimum point. The sensitiveness, too, does not increase, but rather falls off with a more powerful light.

The method I am about to describe is nearly as simple; and, as it depends on the crossing of two Nicols, its sensitiveness is only limited by the power of the source of light. In its simplest form the process is as follows:—An auxiliary Nicol is fixed on the table of the spectrometer, the polarizer turned till the light is quenched, and the reading taken. Then the table of the spectrometer is turned through two right angles—the axis of rotation having been previously set perpendicular to the incident light—the light quenched, and the reading taken again. *The plane of polarization now leans as much to one side of the axis as it did before to the other.* So the mean of the two readings is the reading when the plane of polarization is parallel to the axis of rotation of the table.

The above is of course only a general explanation. The

statement in italics requires fuller examination, and we shall find that it is only strictly true when the Nicol is symmetrical. Let us, then, first suppose that the Nicol is perfectly symmetrical—that is, that the two faces are parallel and the optic axes of the two halves coincident. We shall assume throughout the investigation that the axis of rotation is accurately perpendicular to the incident light. In the previous note we have carefully determined the position of the plane of polarization in terms of the direction of the emergent light, and we have shown that, to a very close approximation, it is at right angles to that position of the plane of polarization of light incident along the same path which gives complete extinction in the Nicol. The discrepancy was shown to be negligible. So there is no ambiguity in speaking of the plane of polarization of either half of the Nicol for a particular direction of the light.

In the first position of our Nicol on the spectrometer, the emergent light is parallel to the incident, and the planes of polarization of the two halves are also parallel. The axis of rotation, too, is at right angles to the incident light. Imagine, then, a line fixed relatively to the Nicol to be drawn from the second face parallel to the emergent light. When the Nicol has been rotated through two right angles, this line becomes again parallel to the incident light, and the plane of polarization of the second half of the Nicol now leans exactly as much to one side of the axis of rotation as the plane of polarization of the first half leant before to the other. The process therefore is so far strictly accurate.

Secondly, let us suppose that the Nicol, though not symmetrical, yet produces no angular deviation on light traversing it. We have seen in the previous note that the plane of polarization of a Nicol is nearly at right angles to the principal plane, and that to a first approximation the small deviation is proportional to the angle between the external light and the principal plane.

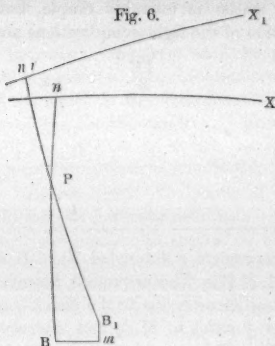
Let Xn , X_1n_1 be the principal planes of the two halves.

B , B_1 their poles.

P the direction of incident and emergent light.

Draw Bm at right angles to BP and B_1m at right angles to B_1m . We may assume that BB_1 is very small and BP

nearly a right angle. The angle between the planes of polarization of the two halves is composed of two parts, of



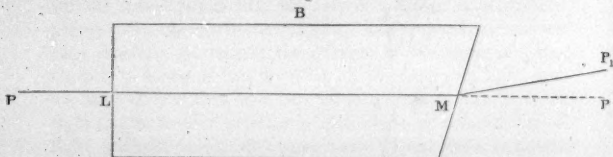
which one, BPB_1 , is approximately equal to Bm , and the other is approximately proportional to the difference between Pn_1 and Pn and therefore to B_1m . So the angle is independent of small variations in the position of P , and is therefore a constant for each particular Nicol. In some Nicols, as we go from one half to the other, the plane of polarization is turned in the direction of a right-handed screw. These we may call right-handed Nicols with a certain rotation-angle. Supposing, then, our auxiliary Nicol is right-handed, the plane of the second half leans at first too much to the right, but, when the Nicol has been rotated through 180° , it leans too much to the left. The correct reading in the case of a right-handed Nicol is therefore that obtained by turning the Nicol circle from the mean reading through half the rotation-angle in the right-handed direction.

The question now arises, how we are to determine the rotation-angle. The following process explains itself. Place the auxiliary Nicol B on the path of the light from the polarizer A , which is mounted in a graduated circle. Turn A till the light is quenched and take a reading. Then turn A so as to allow the light to pass through B , and place in the path of the light from B a third Nicol C . Adjust C till the light

from B is quenched. Remove B, cross C with A, and read again.

So far the matter is tolerably simple, but if the Nicol produce deviation of the light complications are introduced.

Fig. 7.



The figure represents a flat-ended Nicol B traversed by a ray of light $PLMP_1$. The argument, however, is perfectly general. A small error is due to the Nicol C having to deal first with light parallel to MP_1 , and afterwards with light parallel to PP' . This may be eliminated by taking two readings, the Nicol B in the interval having been turned through two right angles or thereabouts about LM. The mean thus found gives us the position of the plane of polarization of PL , which is most nearly parallel to the plane of polarization of MP_1 . We know nothing about the plane of polarization of light which issues parallel to MP' . Clearly therefore, when our Nicol is mounted on the spectrometer-table, we ought not to turn it exactly through 180° , but we ought to turn it till the incident light is related to the face M in the same way as P_1M was at first.

We have then this practical rule. Place the auxiliary Nicol on the spectrometer in such a position that the deviation lies wholly in the plane of rotation. To get the second reading, turn the table in the direction of the deviation through an angle equal to 180° less the deviation. Correct the mean of the two readings by half the rotation-angle as before.

Thus we are able to get an accurate value of the zero-reading in spite of the dissymmetry of the Nicol and the deviation of the light, provided only the two faces of the Nicol are good planes. We have been compelled, however, to limit ourselves to first approximations. On this point it is to be noticed that the outstanding error is due to two causes,

the inaccuracy of adjustment of the auxiliary Nicol, and the dissymmetry of the Nicol itself. The latter we may assume to be very small; while the former has very slight effect. It only comes in as a secondary cause, for if the dissymmetry were removed the outstanding error would be zero. Hence it is not necessary in general to take elaborate precautions about the setting of the Nicol in the determination either of the zero-reading or of the rotation-angle. If great accuracy be required, it will be sufficient to attach permanently to the Nicol a small reflecting surface, and make this always perpendicular to the incident light. The planes of polarization of both halves will then be perfectly definite.

I have treated the method as applied to a Nicol used as polarizer, but it is by no means restricted to this case. The polarizer may be any kind of polarimeter, or if necessary the polarimeter may be the analyzer, while the auxiliary Nicol is used as polarizer. The method occurred to me when I was engaged on some observations on the refraction of polarized light at the surface of Iceland spar. I will describe the arrangements in so far as they bear on the matter in hand; for, as it happened, they gave a very high degree of accuracy in the determination of the zero-reading.

As my source of light I employed part of the filament of a Swan incandescent lamp. I selected a straight piece, and cut off the light from the rest of the filament with a diaphragm placed immediately in front of the lamp. The filament was placed at the principal focus of a lens, from which the light passed to the polarizing Nicol. Then came the spectrometer, from which the collimator had been dismounted; so only the telescope remained. When the telescope was focused for infinity and directed towards the light, a sharply defined image of the selected portion of the filament was seen. The top and bottom limits of the image were not hard lines, owing to the diaphragm not being quite at the principal focus of the collimating lens. They were, however, sufficiently definite for practical purposes.

It was necessary for my other observations that the axis of rotation of the spectrometer should be set at right angles to the light coming from the middle of the filament. I accomplished this in the following manner. I first mounted a

reflecting surface on the table of the spectrometer, and adjusted it to be parallel to the axis of rotation. This may be done with accuracy although the telescope be tilted. Next I adjusted the axis of the telescope to be at right angles to this surface, by getting the cross wires to coincide with their image formed by reflection at the surface. Then I removed the reflecting surface, directed the telescope towards the light, and tilted the spectrometer till the cross wires coincided with the middle of the image.

When the auxiliary Nicol was mounted on the spectrometer-table and the polarizer turned past the crossed position, the image of the filament by no means entirely disappeared; but a patch of nearly complete extinction moved down the image from the top to the bottom. This was of course to be expected, since different positions of the image correspond to different directions through the Nicols, and therefore to different positions of the two planes of polarization. The reading was taken when the patch was halfway down the slit. This method of reading proved to be very sensitive. Without any special care I could get a number of successive readings whose greatest difference was $1'$. This was quite sufficient for my purposes. Indeed my verniers were only graduated to $1'$. But I am convinced that a far higher degree of sensitiveness could be reached if desired, by proceeding on the same lines.

In securing great accuracy two precautions become very important. The source of light should be of uniform brightness. Here a well-made carbon filament is nearly perfect. Again, as much light should reach the eye from the top of the filament as from the bottom, abstraction being made of course of what is stopped by the polarizing properties of the Nicols. For this it is convenient that the lateral throw of the Nicols should be as small as possible. Flat-ended Nicols are therefore to be preferred. But if a pair of Nicols were made for the purpose, it would be easy so to slant the faces that the extraordinary ray should go straight through, and there should be no lateral throw.

The incandescent lamp is a more powerful source of light for this purpose than is perhaps at first sight apparent. For it may be shown that with a given slit and object-glass the

intensity of the illumination of the surface of the first Nicol depends solely on the intrinsic brightness of the source of light. The filament is practically a slit with the source of light brought into contact with it, so there is no difficulty about the whole of the surface of the Nicol being illuminated, and the illumination is just as powerful as would be given by a glowing sheet of the same brightness half an inch broad placed a little distance behind a slit. The intrinsic brightness of the filament of a Swan lamp is many times greater than that of a good gas-flame viewed edgewise. Besides, even if the gas-flame be placed very close to the slit, it is only the nearer portion of the flame that illuminates the whole surface of the Nicol. The further portion only lights up a narrow strip in the middle of the surface.

I used two flat-ended Nicols, belonging to Mr. Glazebrook, of about half an inch aperture. One of these, which gave an angular throw of $15'$, had a left-handed rotation-angle of $38'$.

I find that the plan of reading by means of the motion of the patch of extinction has been the subject of an elaborate paper by Lippich (*Sitzb. der kais. Akad. der Wissensch. Wien*, Febr. 1882).

V. *On some new Forms of Polarizing-Prism.*

By Mr. H. G. MADAN*.

MR. MADAN exhibited and described some new forms of polarizing-prisms. The first of these is by M. Bertrand, and has been described by him (*Comptes Rendus*, September 29, 1884). The prism consists of a parallelopiped of dense flint glass of refractive index 1.658, the same as that of Iceland spar for the ordinary ray. The glass prism is cut like the spar of a Nicol's prism, a cleavage-plate of spar being cemented between the two halves by an organic cement of refractive power slightly greater than 1.658. A beam of light traversing the prism is incident upon the spar at an angle of $76^{\circ} 44'$. The ordinary ray passes through without change; but the extraordinary ray is totally reflected at the

* Read February 28, 1885.

first surface. The prism gives a field of 40° . M. Bertrand's prism has the great advantage of requiring only a very small quantity of Iceland spar, a substance that is becoming very scarce and expensive. The other prisms shown were: a similar one by M. Bertrand, described in the same paper; a double-image prism by Abrens, described in the *Phil. Mag.* for January 1885; and a modification of the latter by Mr. Madan, described in '*Nature*' for February 19.

VI. *The most Economical Potential-difference to employ with Incandescent Lamps.* By Professors W. E. AYRTON, *F.R.S.*, and JOHN PERRY, *M.E.**

THE subject in connection with which the accompanying paper is a small contribution is one of considerable commercial importance. It has long been known that the luminous power of an electric lamp increased much more rapidly than the power electrically expended on it; or, that the number of candles per horse-power increased as the lamp was made to become brighter and brighter. Perhaps the earliest experiments on the subject were those published by Sir William Thomson in 1881, and those made by our students in 1880. Since that time (that is, during the last five years) tests of the efficiency of various types of incandescent lamps have been made over and over again by various persons, without perhaps its being clearly realized that such efficiency-experiments by themselves gave us no idea of the commercial value of any particular lamp.

It is not sufficient to know that when a lamp is giving out a certain number of candles it absorbs so much power per candle, and when giving out a much larger number of candles it absorbs so much less power per candle; but what must be known in addition is the life of the lamp at each of these two candle-powers, before we can decide whether it is more economical to use the lamp with the filament at not much more than a dull red or when brilliantly luminous with a bluish tint.

* Read February 28, 1885.

For with the filament at a comparatively low temperature, although the efficiency of the lamp is low, its life will be great; whereas if the temperature of the filament be high, the large efficiency will be to some extent balanced by its short life and the consequent large cost due to lamp renewals.

In the 'Electrician' for January 31st of this year M. Foussat gives a table of values of the life of a 100-volt Edison lamp for different potential-differences; and M. Foussat has assured us that these numbers were obtained experimentally, and not from calculation according to any theory. That the results should lie so nearly in a curve is undoubtedly at first sight a little striking; but if it be remembered that they are stated to represent the results obtained from the *averages* of a very extended series of tests, their great regularity is not more striking than is a curve showing graphically the result of mortality tables. We asked M. Foussat whether he had also the results of experiments on the efficiency of the same type of 100-volt lamps, as our own experiments on the efficiency of Edison lamps had been made with 55-, 108-, and 110-volt lamps. As, however, he had no such results, and as his tests of lives are the only ones that we now have, and possibly may have, until the completion of the excellent work at present being carried out by the Electric-Lighting Jury of the Health Exhibition, we commenced this investigation by endeavouring to combine the lives given by M. Foussat with the results previously published for efficiency. To enable us to do this, we assumed that the life of 1000 hours given by M. Foussat for his 100-volt lamps when used with a potential-difference of 100 volts would be the same as that for our 108-volt lamp when employed with a potential-difference of 108 volts; or, in other words, we regarded lives given by him as applicable to our 108-volt Edison lamps when each of his potential-differences was multiplied by $\frac{108}{100}$.

To ascertain the most economic potential-differences, we must proceed as follows:—Let $f(v)$ be the life in hours as a function of the number of volts constantly kept on the lamps, $\theta(v)$ the number of candles emitted by one lamp as a function of the potential-difference in volts employed, let p be the price in pounds paid for one lamp, and n the number of

hours per year that a lamp is kept burning ; then

$$\frac{p \times n}{f(v) \times \theta(v)}$$

stands for the cost per year per candle, as far as the renewal of lamps is concerned.

Next, let H stand for the cost in pounds of an electric horse-power per year for the number of hours an electric horse-power is employed. It is often assumed that H is proportional to the number of hours per day that the power is used, or that the yearly bill for power should be based on the horse-power hours, or total energy consumed ; but this idea, which runs even through the Electric Lighting Act, is quite an erroneous one. H will be of the form $h + F(n)$, where h is a constant independent of the number of hours and depending on the rent of the site, capital expended upon engines, dynamos, leads, &c. ; and $F(n)$ is some function of the number of hours during which electric power is required, and depends on the cost of coal, superintendence, &c. If the light were only required for one or two hours in a district where rent was very high, h would be the all-important term and $F(n)$ would be unimportant ; whereas if the electric power were required for various purposes, for, say, 15 or 20 hours out of the 24, in a place where rent was low but coal dear, $F(n)$ would be the important item in the yearly bill for electric power supplied.

Let $\phi(v)$ be the watts per candle, expressed as a function of the number of volts employed at the terminals of a lamp ; then

$$\frac{H}{746} \times \phi(v)$$

represents the cost per year per candle as far as the production of electric power is concerned.

The total cost, therefore, per year per candle is

$$\frac{p \times n}{f(v) \times \theta(v)} + \frac{H}{746} \phi(v) \text{ pounds,} \quad \dots \quad (A)$$

and we must find the value of v that makes this a minimum.

There are two ways in which such a problem can be solved : the one a graphical method, the other an analytical method.

The former may be used by even elementary students, and will be given first. It consists in drawing curves to represent, 1st, $f(v)$ in terms of v , 2nd, $\theta(v)$ in terms of v , and 3rd, $\phi(v)$ in terms of v ; and from these the values of $f(v)$, $\theta(v)$, and $\phi(v)$ are each determined graphically for many values of v , and the value of A calculated for each of these values. A fourth curve is then drawn, connecting the values of A with those of v , when it is easy to see by inspection for what value of v the expression A has a minimum value.

The following is the result so obtained for the 108-volt Edison lamps used for lighting the Finsbury Technical College:—

p is taken at 5s., or £0.25;

n as 560 hours, the time per year, approximately, during which the lamps are lighted;

H as £5: this is perhaps a rather high estimate for the cost of power, considering that the interest on the steam-engine, dynamos, &c., price of coal burnt, wages of the engine-driver and stoker have to be mainly debited to the driving of the College workshops, supplying power for the dynamos worked for experimental purposes; but it will be accurate enough to take the sum of £5 per year as representing yearly interest on extra plant and the yearly interest on the extra cost of supplying one electric horse-power during the 560 hours.

Then the value of v which makes A a minimum turns out to be about 106 volts; and on account of the flatness of the curve connecting the expression A with v , we see that in this particular case the annual cost of supplying light is only increased by 3.5 per cent., if the potential-difference between the mains be kept diminished to about 104.8 or kept up to about 108 volts. Also, that keeping the potential-difference diminished to about 104.5, or increased to about 108.5 volts, increases our total annual cost of lighting by 5 per cent.

More recently one of our students, Mr. Robertson, has been making efficiency-experiments on some Edison lamps obtained from France; and by trial he has found some that give 16 candles at about 100 volts, and are therefore presumably of the same type as those employed in the life-

experiments given by M. Foussat in the 'Electrician' for January 31st.

Mr. Robertson's results from one such lamp are as follow:—

TABLE I.

Candles, or $\theta(v)$.	Volts, or v .	Watts.
3	84	56.62
4	87	63.51
5	87.8	66.03
6	90.5	69.05
7	93.5	75.74
10	96.75	79.34
11	97.61	81.01
12	98.04	82.35
14	98.9	85.55
16	100.6	90.54
18	101.5	92.37
20	103.6	96.66
22	105.4	100.10
24	107.5	104.28
26	109.2	108.11
28	110	111.10
30	111.4	113.63
35	114	118.56
40	115.2	122.11

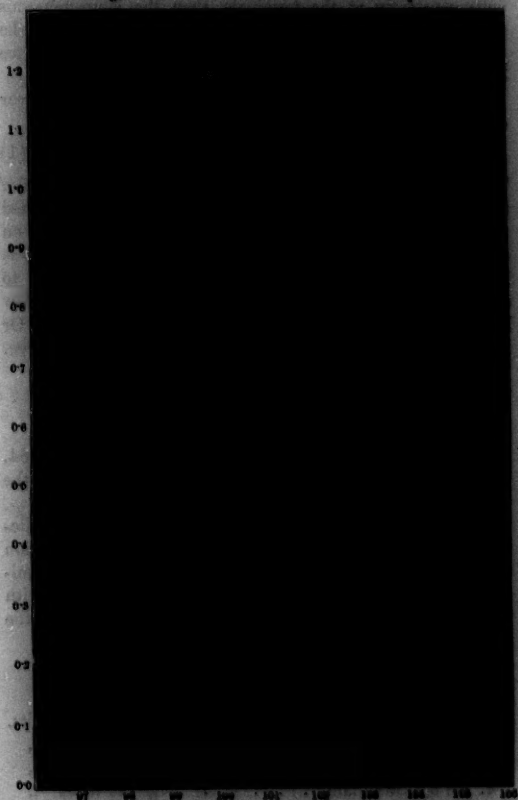
We are therefore now in a position to take up the problem in a more direct manner, without making any assumption beyond this—that Mr. Robertson's 16-candle 100-volt French Edison lamp is of the same kind as M. Foussat's 16-candle 100-volt French Edison lamp.

Solving the problem graphically, in the way previously described, and using the same values of p , n , and H —viz. cost of a lamp 5s., number of hours of burning per year 560, and annual cost of an electric horse-power for those 560 hours £5,—we obtain the curve AAA to represent the cost per candle per year as regards renewal of lamps, BBB the cost per candle per year as regards power, and the resultant curve CCC as the total cost per candle per year.

The minimum value of this cost appears from the curve CCC to be at about 101.4 volts, and to equal about 11d. per candle per year. If the potential-difference be maintained constantly down at 98.7 volts, or up at 104 volts, then the cost becomes 1s. per candle per year. It will also be seen

from the curves that the yearly cost for renewals of lamps is but a small fraction of the total yearly cost, as long as we are

Diagram for a 16-candle 100-volt Edison Lamp.



using potential-differences of not more than about 100 volts; so that when using, with these lamps, this or a lower potential-

difference, the actual sum paid for the lamps is not so very serious an item in the yearly lighting bill. But if, on the other hand, we maintain a potential-difference of, say, 104 volts constantly at the terminals, then the lamp renewals represent, in our particular case, nearly one third of the yearly expenditure, and consequently any change in the price of lamps becomes extremely important.

We find that the simplest expression for the candle-power of this incandescent lamp in terms of volts is

$$\theta(v) = a(v-b)^3. \quad \dots \dots \dots (1)$$

Thus, for example, it will be found that if the cube roots of the values of $\theta(v)$, given in the preceding table, are plotted with the corresponding values of v as coordinates of points on squared paper, the points lie in no regular curve so nearly as in a straight line, discrepancies being apparently due to errors of observation, and these errors of observation follow a periodic law which may be of interest physiologically. Or it may be that there really is a point of inflexion in the curve.

For the particular lamp in question we find that

$$a = 0.0002621,$$

$$b = 62.12;$$

or, if the law remains true for a less number of volts than 84, the lowest used in this experiment, then when $v = 62.12$, the candle-power is zero.

We do not, however, find that the cube roots of the candle-powers of all sorts of incandescent lamps follow a line-function of the difference of potential at their terminals; for on examining the results of experiments that have been made with various types of lamps, we find that with the first five of the following set of lamps the law

$$\sqrt[3]{\theta(v)} \propto v - b$$

is very nearly true.

1. A Lane-Fox lamp from 50 to 80 volts.
2. A British Electric Light Co.'s lamp " 50 to 68 "
3. An Edison "7 B" 8-candle lamp . " 45 to 63 "
4. An old Swan lamp " 40 to 64 "
5. A low-resistance 8-candle Swan lamp " 16 to 40 "

Whereas with the next five lamps it is not so true, although

for half the range the law might be regarded as true in all the cases.

6. A Maxim lamp from 42 to 57 volts.
7. A Maxim lamp „ 35 to 110 „
8. An old form Swan lamp „ 25 to 54 „
9. A high-resistance Swan lamp . . „ 45 to 90 „
10. An Edison B 8-candle lamp . . „ 36 to 85 „

We may here mention that we have found the following plan very useful in helping us to draw correctly a curve at a place where there is a sudden bend and an absence of points, determined from experiments to guide us in drawing the curve. Instead of plotting x and y , and obtaining a curve which it would be very difficult to draw correctly, we may by using some simple function of y obtain points which obviously lie in a curve which it is easy to draw. Thus we may plot $\sqrt{y}$ and x , or $\sqrt[3]{y}$ and x , or $\log y$ and x . When the curve connecting y and x is not roughly asymptotic to the axis of x , but to some line parallel to this axis, it is obvious that there is a greater likelihood of obtaining simple curves by plotting $\sqrt{y+\alpha}$ and x , or $\sqrt[3]{y+\alpha}$ and x , or $\log(y+\alpha)$ and x , where α is some constant obtainable by inspection, than by simply plotting the function of y alone.

The following table gives M. Foussat's lives in terms of v and the corresponding values which we have calculated of $\log f(v)$, $\log \theta(v)$, and of $\log f(v)\theta(v)$ from the values of $\theta(v)$ given in the previous table.

TABLE II.

v .	$f(v)$.	$\log f(v)$.	$\log \theta(v)$.	$\log f(v)\theta(v)$.
95	3595	3.5557	0.9729	4.5286
96	2751	3.4395	1.0119	4.4514
97	2135	3.3294	1.0485	4.3779
98	1645	3.2161	1.0851	4.3012
99	1277	3.1062	1.1199	4.2261
100	1000	3.0000	1.1556	4.1556
101	785	2.8949	1.1898	4.0847
102	601	2.7789	1.2216	4.0005
103	477	2.6785	1.2531	3.9316
104	375	2.5740	1.2852	3.8592
105	284	2.4533	1.3152	3.7685

When $\log f(v)$, $\theta(v)$, and v are plotted as coordinates of

points, these points are found to lie so nearly on a straight line that the formula

$$\frac{1}{f(v)\theta(v)} = 10^{0.07545v - 11.097} \quad \dots \quad (2)$$

is found to be true with considerable accuracy, and such a formula lends itself with great ease to calculation.

It is quite true that we might have obtained a still more accurate formula than (2) since, as published by Mr. Wright, one of the members of our class, in the 'Electrician' for February 21st, the logarithm of the life of a lamp is shown to be a line-function of the difference of potentials at which it is worked.

We may in fact with this particular type of lamp put

$$f(v) = 10^{14 - 0.11v} \quad \dots \quad (3)$$

with very great accuracy indeed.

Combining this with (1), we obtain as a more accurate formula,

$$\frac{1}{f(v)\theta(v)} = \frac{1}{a} 10^{0.11v - 14} (v - b)^{-3} \quad \dots \quad (4)$$

But unfortunately this more accurate formula does not lend itself to mathematical calculation, whereas that given in (2) is very suitable for this, and has a sufficient degree of accuracy for our purpose. In using (2) we are really using for the candle-power

$$\theta(v) = 10^{0.03456v - 2.303} \quad \dots \quad (5)$$

instead of (1).

It will be also found that from 95 to 105 volts, the range of volts given in Table II., the values of $\phi(v)$, or watts per candle-power, when corrected for errors of observation, satisfy with considerable accuracy the equation

$$\phi(v) = 3.7 + 10^{8.007 - 0.07667v} \quad \dots \quad (6)$$

If, however, we take the whole range of values given in Table I., then it will be found that the equation

$$\phi(v) = 2 + 10^{4.424 - 0.03793v} \quad \dots \quad (7)$$

is better satisfied than (6).

The following Table III. gives the numbers we have actually employed in making these calculations.

TABLE III.

<i>v</i> .	$\sqrt[3]{\theta(v)}$. corrected.	$\theta(v)$ corrected.	Watts corrected.	Watts per candle-power $\phi(v)$.	$\log \{\phi(v)-2\}$.
84	1.4	2.744	56	20.408	1.2650
86	1.53	3.582	60.15	16.79	1.1700
88	1.66	4.574	64.25	14.27	1.0888
90	1.786	5.7	68.4	12.0	1.0000
92	1.914	7.0	72.6	10.37	.9227
94	2.044	8.55	76.85	9.0	.8451
96	2.175	10.28	81	7.882	.7695
98	2.3	12.167	85.15	7.0	.6990
100	2.428	14.24	89.3	6.271	.6305
102	2.556	16.69	93.5	5.6	.5663
104	2.682	19.29	97.8	5.07	.4871
106	2.81	22.188	101.9	4.593	.4138
108	2.94	25.412	106	4.172	.3369
110	3.065	28.79	110.1	3.824	.2610
112	3.195	32.61	114.25	3.504	.1772
114	3.32	36.594	118.3	3.233	.0910

Using (6) and (2), the total cost of one candle per year is

$$p n 10^{0.07545 v - 11.687} + \frac{H}{746} (3.7 + 10^{8.007 - 0.07867 v}).$$

This is a minimum when

$$v = 110.66 + 6.574 \log \frac{H}{p n}.$$

Hence taking $p = 0.25$, $n = 560$, $H = £5$, we find

$$v = 101.15 \text{ volts.}$$

Using (7) and (2), we should find in the same way that the minimum is obtained when

$$v = 101.46 \text{ volts;}$$

and these minimum values of v agree very closely with that previously determined graphically.

One very important problem in connection with incandescent lamps, and one that cannot yet be regarded as solved, is the determination of the life of a lamp for any given number of volts, from experiments made either on the efficiencies at several different potentials, or from experiments on the life made at so high a potential-difference that the life will be short, and the experiment made therefore in a comparatively short time.

If, however, an expression of the form

$$f(v) = 10^{a-bv}$$

can be regarded as representing with sufficient accuracy the law of life for all types of incandescent lamps, then if sufficient

experiments be made with a number of lamps at each of two different potential-differences to enable us to determine the average life at each of these potential-differences, the constants a and b can be calculated from the equation, and hence the value of $f(v)$, the life, calculated for any other potential-difference.

In connection with this investigation we have endeavoured, with the aid of one of our Assistants, Mr. Walmsley, to ascertain whether some form of ordinary cheap candle could be used, at any rate for rough photometric measurements, in place of the much dearer standard-candles, and, as far as the following results obtained with seven candles selected at random from a packet of No. 8 sperm-candles go, it would seem that these candles do not differ so very much more in intensity from one another than standard candles are said to do. Of course many more experiments on this subject must be made before the possibility of using cheap candles as a rough standard can be decided on, but in the meantime the following experiments may be interesting.

This particular type of candle, No. 8 sperm, and costing 11d. per pound, was selected, because such candles were found to resemble in thickness the standard candles that we have been accustomed to use, and which cost 2s. 9d. per pound.

TABLE IV.

Name of candle.	Grammes of wax burnt per hour.	Light in terms of that emitted by the Standard candle.
Standard candle ...	7.82	1.00
1st Sperm " ...	Doubtful	1.14 (?)
2nd " " ...	6.102	1.00
3rd " " ...	7.188	1.00
4th " " ...	7.29	1.02
5th " " ...	7.1	1.02
6th " " ...	6.84	1.05
7th " " ...	6.66	0.99

These tests of candle-power were not made with any very high degree of accuracy; but the comparison of these No. 8 sperm-candles with the standard was carried out with probably quite as much accuracy as is employed in making ordinary commercial experiments on the luminosity of incandescent lamps.

VII. *On the Second Law of Thermodynamics.*

By Mr. J. MACFARLANE GRAY*.

MR. MACFARLANE GRAY gave an account of a most extended investigation upon the second law of thermodynamics. From considerations connected with the specific heats of liquids and gases, the author comes to the conclusion that the second law is not true. The experimental results used are chiefly those of Regnault, to which, however, Mr. Gray has applied some corrections.

VIII. *Lecture-Experiments on Spectrum Analysis.*

By E. CLEMINSHAW, M.A., F.C.S.†

I PROPOSE in the following paper to explain some processes by which all the phenomena in spectrum analysis, usually shown upon the screen in class-demonstrations, may be exhibited without the use of the electric light, which for various reasons is frequently not available. The methods employed depend upon the use of the limelight and the oxy-hydrogen flame : these of course do not give the same brilliant effects as the electric light, but all the phenomena may be shown upon the screen with much simpler apparatus ; and for some of my methods I think I may claim advantages over the usual method of demonstration.

i. *Spectra of the Alkalis and Alkaline Earths.*

Debray (*Ann. Ch. Phys.* [3] lxx. p. 331) proposed the use of the oxy-hydrogen flame for the volatilization of the alkalis and alkaline earths. By a modification of the ingenious method of Bunsen for procuring a monochromatic flame, the substances in question may be conveniently introduced into the flame. Bunsen made hydrogen in a bottle containing a strong solution of sodium chloride with zinc and hydrogen sulphate,

* Read February 28, 1885.

† Read March 14, 1885.

passed coal-gas into the bottle in order to increase the size of the flame, and to assist in the mechanical carrying-over of the spiritings of the sodium-chloride solution, and burnt the mixed gases in a description of Bunsen burner.

Instead of burning the mixed gases in air, I propose to burn them in an atmosphere of oxygen, by which I find a sufficiently bright flame can be obtained to show the spectra on the screen.

The mixed gases are burnt from a jet consisting of two concentric brass tubes, the inner of which passes into the gas-generating apparatus, and the outer is connected with a supply of oxygen; the inner tube is from $\frac{1}{8}$ – $\frac{3}{16}$ inch diameter, and may with advantage be wider at the bottom. A better result is obtained by passing hydrogen into the bottle, since it is not easy to render the coal-gas perfectly non-luminous by the oxygen.

A rapid evolution of hydrogen is necessary for a good result. The lens or lenses should be arranged so as to obtain as bright an image of the slit as possible, even at the expense of some spherical aberration.

The light obtained with sodium chloride is perfectly monochromatic and very bright. I venture to recommend this method of obtaining a bright monochromatic flame for lecture-demonstrations.

With a strong solution of lithium chloride the red band (Li α) can be shown, and on adding some strong solution of sodium chloride the yellow band of sodium.

For showing the spectra of the alkaline earths I use a saturated solution of the chlorides and generate hydrogen with zinc and hydrogen chloride. With two CS_2 -prism bottles a red, orange (Sr α), and blue (Sr δ) can be shown on the screen; and, if the slit is narrow, the yellow band of sodium is clearly shown as soon as some sodium-chloride solution is introduced into the gas-generator. If a strong solution of lithium, sodium, and calcium chlorides are used, with two CS_2 -prisms five bands can be shown on the screen, including the orange, green, and violet bands of Ca.

The above method is applicable for all compounds which can be volatilized in the oxy-hydrogen flame, and which are not reduced by the nascent hydrogen.

If hydrogen gas is passed into the bottle, or sufficient oxygen used to destroy the luminosity of the coal-gas, no trace of a continuous spectrum can be seen. An occasional addition of acid, to maintain the evolution of hydrogen, keeps the flame burning brilliantly for some time; this method might, therefore, be useful for spectroscopic observations for comparison.

ii. *Continuous Spectrum showing Bright Bands.*

If the slit is narrow, the orange and green bands of calcium may often be seen on the spectrum obtained with the lime-light. If the chlorides of the alkalis or alkaline earths are previously melted upon the surface of the lime-cylinder, the bands of these compounds become very bright. Four different compounds might be melted upon different portions, and the spectra of each shown in turn by turning the cylinder. The lime-cylinder should be a "hardlime;" the softer cylinders are too porous.

iii. *Reversed Spectra.*

The important fact that the light emitted by incandescent sodium vapour is opaque to light from the same source, is easily shown in a striking manner upon the screen by placing the apparatus used for showing the yellow band of sodium in § i. in the centre of an ordinary optical lantern; a bright disk of monochromatic sodium-light may be obtained on the screen from 4-5 feet in diameter. A small Bunsen burner is then placed as near as possible to the condensing lens, with the top of the tube projecting $\frac{1}{2}$ inch above the edge of the lens, which shows the shadow of the tube at the top of the disk upon the screen. When a platinum wire containing a quantity of sodium chloride is introduced into the flame of the Bunsen burner, the dark appearance of the cooler flame is clearly seen on the screen.

Reversal of D line.

Mr. Lewis Wright states that this can be done with the limelight by burning sodium in a Bunsen burner placed as close to the slit as possible. I have not been able to obtain

very satisfactory results by this method ; the flame is too hot. I have been more successful with the following methods :— The sodium is burnt in a small spoon placed between the lens and prism, *just* below the focus of the rays from the lens, which must be made as small as possible, so that all the rays are made to pass through the flame where the sodium vapour is densest, and none of the light from the sodium vapour is focused on the screen, and a much smaller flame may be used. Or the spoon may be placed a little in front of or beyond the focus in the path of the rays ; a shadow of the spoon is seen across the spectrum, and with a small flame a marked thickening of the dark line is seen just above the spoon. The sodium may be burnt in a Bunsen burner or in the flame of a spirit-lamp.

If the sodium-flame is placed between the slit and the lime-light, I have obtained a good result by using a Bunsen burner, the flame of which is cooled down to a proper temperature by a mixture of air and carbon dioxide. Care must be taken not to pass carbon dioxide into the flame to excess, otherwise too great a lowering of its temperature takes place. I have, for instance, obtained in this manner a bluish flame which did not show the slightest trace of sodium in a room where sufficient sodium had been burnt to make every gas-flame give a strong sodium reaction.

The supply of carbon dioxide to the Bunsen burner may be adjusted in the following manner :—A cork is attached to the movable cover which closes the two holes for admitting air, and two holes made in it opposite the air-holes ; to one a glass tube is attached which is connected with a bottle, into which carbon dioxide is passed, fitted with three openings. By opening clamps the carbon dioxide may be all passed into the Bunsen burner, or passed directly out from the bottle. The sodium is first brought into vivid combustion, and then the flame cooled down by admission of the proper supply of carbon dioxide.

The reversed line may also be shown by burning sodium in a spirit-lamp with four wicks, in the centre of which is a jet for the admission of oxygen. This is placed between the slit and the limelight. On passing oxygen into the flame, the heat may be raised sufficiently high to produce a bright

Na band upon the screen, especially if the light from the incandescent lime is somewhat moderated, and turned into a dark band when the oxygen is shut off, proving that the production of a dark or bright sodium-band depends upon the temperature of the absorbent vapour.

IX. *On the Characteristic Curves and Surfaces of Incandescence Lamps.* By J. A. FLEMING, M.A., D.Sc. (Lond.), Fellow of St. John's College, Cambridge*.

RECENT issues of a scientific journal† have contained some interesting letters and notes on the life of incandescence lamps, and on resulting deductions to be made therefrom. The difficulty of the full and complete discussion of this subject is the absence of sufficiently prolonged experiments to give statistics reliable for this purpose. These can only be obtained at great expense and by experiments lasting over a considerable time; but the results to hand make a preliminary investigation interesting on the connection which exists between the life of incandescence lamps and other correlated quantities.

The manufacture of incandescence lamps has now advanced to such a condition that the accidents of manufacture are greatly under control. The conditions necessary to get a good lamp are fairly well understood, and the physical actions going on in the lamp are also to a great extent known. We now know that the expectations of earlier investigators of getting an absolutely unalterable carbon incandescence lamp are not destined to be fulfilled; but we know that the gradual destruction of the filament is an operation dependent upon several causes, which may be greatly delayed by attention to, and success in, certain operations of manufacture.

The gradual destruction of the carbon filament in a vacuum lamp is a kind of erosion taking place at one or more points. Observation seems to show that in a single loop-filament this cutting through takes place most generally near the negative

* Read March 14, 1885.

† 'The Electrician,' vol. xiv. (1885), pp. 246, 294, 311, 347.

side. The determining cause of breakage is, however, temperature; and carbon filaments which present such inequalities of resistance as to give rise to spots of higher temperature might, other things being equal, be expected to be doomed to a short career. Lamp-filaments are therefore like human lives: some come into the world with a taint of disease upon them, in the shape of irregularity of structure, which predisposes to an early death; but nevertheless, as even in the case of suicide, the great law of averages overrides particular instances, and gives us, in the case of a sufficiently extended series of observations, a law connecting the average behaviour under fixed circumstances.

Experience gained during three years of commercial manufacture and use of incandescence lamps has demonstrated that when a large number of filaments are prepared with identical care, and the lamps made with them sorted out into batches, and worked with varying electromotive force, there is a very constant relation between the average efficiency or candles per horse-power and the average duration or life, and the working potential or volts of the lamp. Now, in a general way we do not know what the form of the function is that connects these three quantities, or any two of them, but there have been a large number of observations on the relation of certain variables which indicate as a most probable form an exponential function.

There are four variables between which a relation is required—electromotive force, resistance, candle-power, and life. The third of these is at present somewhat vague and indeterminate. What we really are concerned with is the total eye-affecting radiation; and our present methods only allow of a certain more or less imperfect comparison of this as a whole with that of a standard candle, or a calculation of the integral deduced from observation of the relative intensities of certain rays. Accordingly, of these four quantities two alone can be measured with any great accuracy. One is merely an average, and the other is necessarily a somewhat ill-defined quantity.

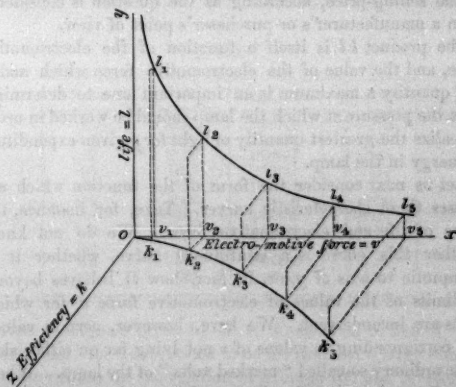
Between any two of these variables we can seek a relation, and having a number of observations we plot down these in what are best called the characteristic curves of the lamp. Now two of the most important of these curves are those

connecting the electromotive force and efficiency, or candles per horse-power, and the electromotive force and life; and they may therefore be called the principal characteristic curves of the lamp. Three other useful curves may be obtained by plotting down the curves connecting candle-power and current, candle-power and electromotive force, and electromotive force and resistance. These may be called subsidiary characteristic curves.

Since the life and efficiency of a lamp vary together with the electromotive force, we can only properly represent the relation between the three by a *surface*, which may be called the characteristic surface of the lamp.

Take three rectangular axes, x, y, z (fig. 1), and let distances measured outwards represent life, candles per horse-power, and electromotive force. Let a curve be drawn on the $y x$ plane, representing the relation of electromotive force and life, and one on the $x z$ plane representing electromotive force and efficiency; let the ordinates be drawn at various

Fig. 1. Diagram of the principal Characteristic Curves of a Lamp.



points to both these curves, starting from abscissæ, representing certain pressures v_1, v_2 , &c. Complete the rectangles on the ordinates k, l , &c.; and we see that these rectangles form the

orthogonal sections of a solid bounded respectively by the two planes xy and xz , and two curved surfaces, of which the characteristic curves l and k are the traces on these planes. The surface of this volume may be called the characteristic surface of the lamp. We see that the area of the orthogonal section parallel to yz gradually increases to a maximum, and then decreases. This is obviously because, for zero electromotive force, life is infinite and candles per horse-power zero; whilst for very high electromotive force, life is zero and candles per horse-power or efficiency a maximum. Now the area of cross section which is a maximum for a certain value v , of electromotive force represents the product of life and candles per horse-power, or the maximum candle-hours per horse-power it is possible to get; and the value of kl is therefore a very important quantity. I shall call this maximum value of kl the principal modulus of the lamp, because the value of the lamp for commercial purposes is obtained by dividing the numeric representing this principal modulus by the price of the lamp, taking either the cost of manufacture or the selling-price, according as the question is considered from a manufacturer's or purchaser's point of view.

The product kl is itself a function of the electromotive force, and the value of the electromotive force which makes this quantity a maximum is an important one to determine. It is the pressure at which the lamp should be worked in order to realize the greatest quantity of light for a given expenditure of energy in the lamp.

Let us next consider the form of the function which expresses these characteristic curves. Take, for instance, the curve of life and electromotive force. We do not know whether this curve is a continuous curve, whether it is asymptotic to axis of y , or, in fact, how it behaves beyond the limits of the values of electromotive force v , for which lamps are incandescent. We have, however, certain values for l corresponding to values of v not lying far on either side of the ordinary so-called "marked volts" of the lamp. When the writer was in America he was shown a large number of observations, made with the view of determining the relation between average life and electromotive force, not far on either side of 100 volts, taken for the Edison 108-volt A lamp. For

this lamp the life-pressure curve at or near the working-pressure is approximately a logarithmic curve whose equation is

$$l = Av^{-\alpha},$$

where A and α are constants; and for the Edison 16-candle 105-volt lamp α is nearly 25; and, accordingly, life varies inversely as the 25th power, roughly, of the electromotive force. In the figures shown to the writer the life-pressure curve had been drawn for pressures corresponding to lives far in excess of what could actually have been observed during the time filament-lamps have been made, reaching up to 11,793 hours. This is obviously improper: we do not know that α is not itself a function of v , and for pressures departing far from the ordinary working-pressure the variation of average life and pressure is not probably expressed by so simple a relation; and we cannot go fairly beyond the limits of actually observed lives. M. Foussat has recently communicated to a scientific journal the results of observations on the life of French Edison lamps.

TABLE I.

Volts.	Life.	$\log_{10}$ (volts).	$\log_{10}$ (life).
95	3595	1.97772	3.55570
96	2751	1.98227	3.43949
97	2135	1.98677	3.32940
98	1645	1.99123	3.21617
99	1277	1.99564	3.10619
100	1000	2.00000	3.00000
101	785	2.00432	2.89487
102	601	2.00860	2.77887
103	477	2.01284	2.67852
104	375	2.01703	2.57403
105	284	2.02119	2.45332

Now the figures given by M. G. Foussat on p. 246 of 'The Electrician' for 1885, stated to be the result of a large number of observations on the relation of average life and pressure at or near 100 volts, conform approximately to the above law. Taking M. Foussat's numbers for life and electromotive force, and taking logarithms of both, we have the figures given in Table I.

Writing down the differences between successive values of $\log v$ and $\log l$, we get the numbers—

455	11621	432	10513
450	11009	428	11600
446	11323	424	10035
441	10998	419	10449
436	10619	416	12071

Now, if $l = Av^{-\alpha}$, then

$$\log l - \log l_1 = -\alpha(\log Av - \log Av_1),$$

where l and l_1 , v and v_1 are adjacent values of l and v ; accordingly,

$$\alpha = -\frac{\log l - \log l_1}{\log Av - \log Av_1}.$$

The mean of the differences of $\log l$ divided by the mean of the differences of $\log v$ gives 25.4 nearly; and, accordingly, M. Foussat's numbers agree with those found in America in assigning to α a value not far from 25 for the Edison lamp. In the neighbourhood of 100 volts

$$l = Av^{-25};$$

or, average life varies inversely as the 25th power of electro-motive force.

This, however, is easily seen to be a very rough approximation. The successive quotients of life-difference by volt-difference are not constant; and the above simple exponential formula cannot be admitted as anything more than a very imperfect connection. In examining the Carlisle Tables of Mortality, which give the expectation of life at every age drawn from a very large number of observations, it was apparent that an empirical formula connecting the two quantities could be obtained of a form

$$\log e = a + bx + cx^2 + \&c.*,$$

where e = expectation of life at any age x .

I was led therefore to try if such an empirical formula

* Let x be the age, and e the expectation of life at that age. Then by the Carlisle Tables of Mortality at the several ages 10, 20, 30, 40, 50, 60, 70, 80, 90, the corresponding expectations of life are:—48.82, 41.46, 34.84,

better suited the present case; and a process of trial and failure showed that

$$10 \log l = 135 - v - \frac{v^2}{2000}$$

is a formula which gives very nearly correct results in calculating the life of a lamp l , given the working-pressure v in volts. This may be expressed otherwise:—

$$\log l = 13.5 - \frac{v}{10} - \frac{v^2}{20,000},$$

or

$$l = 10^{13.5 - \frac{v}{10} - \frac{v^2}{20,000}}.$$

Calculating by this formula, we get the following values for $\log l$ and l , l being the average life in hours of the 100-volt Edison lamp as made in France:—

TABLE II.

Calculated by formula.		Observed.	
l .	$\log l$.	$\log l$.	l .
3539	3.5488	3.5557	3595
2749	3.4392	3.4395	2751
2136	3.3296	3.3294	2135
1658	3.2196	3.2167	1645
1289	3.1100	3.1082	1277
1000	3.0000	3.0000	1000
776	2.8900	2.8948	785
602	2.7798	2.7788	601
467	2.6698	2.6785	477
362	2.5592	2.5740	375
281	2.4488	2.4533	284

It is evident, then, that the simple exponential function does not give nearly so good results as a formula of this latter description; and there is no doubt but that by a suitable selection of constants a formula can be obtained expressing the life of the lamp as a function of electromotive force throughout an observed range, which shall be closely in

27.61, 21.11, 14.34, 9.18, 5.51, 3.28; and it can easily be found that

$$10 \log e = 17.2 - \frac{x}{100} - \frac{14}{10,000} x^2, \text{ very nearly;}$$

the calculated values from which are:—49.6, 44, 36.6, 28.6, 20.9, 14.3, 9.64, 5.54, 3.14.

accordance with observed facts*. Until, however, a much larger collection of statistics is obtained we shall not be in any position to determine if these constants are definite for each type of lamp or kind of carbon, and whether the average life of a lamp can be predicted from a knowledge of these constants.

An attempt was made, in the next place, to endeavour to obtain an approximate empirical formula connecting the efficiency of a lamp, or the candles per horse-power, and the electromotive force. On April 13, 1882, Prof. A. Jamieson read a paper before the Society of Telegraph Engineers and of Electricians (*Journ. Soc. Tel. Eng.* vol. xi. p. 164), "On Tests of Incandescent Lamps," and he has there given a number of tables and curves for different lamps, giving the efficiencies and resistances for various electromotive forces.

These observations afford a convenient means of putting to the test empirical formulæ, because the observations seem to have been carried out with very great care, and being done with secondary and primary batteries as current generators, the observations are more likely to be accurate than when a current from a dynamo machine is used; also because the observations for candle-power were entrusted to Dr. Wallace, gas-analyst for Glasgow, and were therefore in the hands of an observer whose eye was probably more trained to detect minute differences of illumination than one not so familiar with such work.

Professor Jamieson gives one complete table of the constants of an Edison 8-candle lamp over a great range of candle-power. Selecting that portion of the table in which the electromotive force was high enough to illuminate the lamp, we have as follows:—

* Mr. F. M. Wright has given, in 'The Electrician,' p. 311 (1885), a formula—

$$100 - V = 9.098213 \log \frac{L}{1000}$$

This is equivalent to $9.00 \log L = 127 - V$;

and Professors Ayrton and Perry have given

$$L = 10^{14 - 11v}$$

or

$$10 \log L = 140 - 11v,$$

both of which are nearly equivalent to the formula given in the text.

TABLE III.

Tests of an Edison 8-candle Lamp, made by Prof. Jamieson,
March 8, 1882.

Resistance, R.	E.M.F. in volts, V.	Current, in amperes, A.	Candle-power, K.
63.7	45.9	0.722	5.2
63.3	46.7	0.737	6.2
62.7	48.3	0.77	8.2
61	51.9	0.85	12.9
60.6	53	0.874	14.3
59.3	56.2	0.948	21.3
58.4	58	0.995	25.3
57.8	61.1	1.06	35.8
57	63.1	1.21	43.8

If we take the logarithms of these numbers we have the following table :—

Edison 8-candle lamp.

log R.	log V.	log A.	log K.
1.80414	1.66181	1.85854	0.71600
1.80140	1.66932	1.86747	0.79239
1.79729	1.68395	1.88649	0.91381
1.78533	1.71517	1.92942	1.11059
1.78247	1.72428	1.94151	1.15534
1.77305	1.74974	1.97681	1.32838
1.76641	1.76343	1.99782	1.40312
1.76193	1.78604	0.02531	1.55388
1.75587	1.80003	0.08279	1.64147

Let E be the watts of the lamp and k the efficiency or candles per horse-power. Then

$$k = \frac{K 746}{\Delta V};$$

and if we calculate the logarithms of the efficiency corresponding to each electromotive-force value, and compare these with four times the value of the corresponding electromotive force, we have the following table :—

$\log k.$	$4 \log v.$	$4 \log v - \log k.$
2.06839	6.64724	4.57885
2.13834	6.67728	4.53894
2.21611	6.73580	4.51969
2.33879	6.86068	4.52189
2.36229	6.89712	4.53483
2.47457	6.99896	4.52439
2.51461	7.05372	4.53911
2.61527	7.14416	4.52889
2.63137	7.20012	4.56885

Excepting the first and last values, which are the result of observations on the candle-power at extreme values, the intermediate figures are not very far from constant, and indicate, as a first rough approximation, that efficiency varies as the fourth power of the electromotive force.

At both high and low candle-powers the comparison of the light with a standard candle is difficult. In one case an excess of red, and in the other an excess of violet rays makes the comparison of the naked lights much more difficult at extremes, and, as is well known, the efficiency for very high or low candle-power must be stated for a definite radiation. The only other comparison between these two variables attempted has been in the case of the experiments made by the Committee appointed to report on the incandescence lamps in the Paris Electrical Exhibition, an abstract of which appears in the same volume of the 'Journal of the Society of Telegraph Engineers.' The Committee condense these results on four varieties of lamps into the following numbers for the efficiencies, or candles per horse-power and electromotive force working them, taken at two very different values.

	Edison.		Swan.	
E.M.F.	89.11	98.39	47.3	54.21
Candles per horse-power ...	196.4	307.2	177.9	262.5
	Lane-Fox.		Maxim.	
E.M.F.	43.63	48.22	56.49	62.27
Candles per horse-power ...	173.6	276.9	151.3	239.4

If we take the logarithm of each number, and compare the difference of the logarithm of the efficiencies with the differ-

ence of the logarithm of the corresponding electromotive forces for each lamp, we have the following ratios :—

Edison.	Swan.
$\frac{19428}{4302} = 4.5,$	$\frac{16895}{5922} = 2.9.$
Lane-Fox.	Maxim.
$\frac{20277}{4344} = 4.6,$	$\frac{19928}{4231} = 4.7.$

If, therefore, k_1 and k_2 be the efficiencies corresponding to two observed values v_1 and v_2 of electromotive force,

$$\frac{\log k_1 - \log k_2}{\log v_1 - \log v_2} = 4.5 ;$$

or

$$k = Cv^{4.5}$$

represents an approximate formula for calculating the efficiency. If, now, approximately, in the case of an Edison lamp, the candles per horse-power vary as the 4.5 power of the electromotive force, and the life varies inversely as the 25th power of the electromotive force, it follows that

$$l \propto \frac{1}{k^{5.7}}$$

or, roughly, that the life varies inversely as some power between 5 and 6 of the candles per horse-power.

The writer was given in America an extensive series of observations on the relation between the candles per horse-power and the average life, which were as follows :—

Life.	Candles per H. P.
11793	100
7150	110
4528	120
2974	130
2015	140
1403	150
1000	160
727	170
539	180
406	190
310	200
236	210
102	220

The length of life, 11,793 hours, given as corresponding to an efficiency of 100 candles per horse-power, is nearly equal to four years of average burning in working hours, and could not have been the result of actual observation; but these numbers agree very closely with the law that

$$l \propto \frac{1}{k^{5.25}},$$

where l represents life in hours and k candles per horse-power. It is most probable that this law has been deduced from observations on the life lying within an observed range and then extended by calculation to efficiencies below those actually observed. In any case these observations are not in great discord with the above deductions made from the numbers furnished by M. Foussat on the connection between life and electromotive force, in conjunction with other observations on the relation of efficiency to working pressure in the case of Edison lamps.

It is, however, far more probable that the connection between the candles per horse-power and the working pressure is expressible by a formula of this kind,

$$\log k = \alpha + \beta v + \gamma v^2 + \&c.,$$

where α, β, γ are known constants; and in this case we should have then both the life-pressure characteristic curve, and the efficiency-pressure characteristic curve expressed by analogous equations:—

$$\log l = a + bv + cv^2 + \&c.,$$

$$\log k = \alpha + \beta v + \gamma v^2 + \&c.$$

We have seen above that such a formula does fit in with observed values for one pair of variables. Further examination of this point is desirable.

Professors Ayrton and Perry have drawn attention* to a connection between the candle-power of a lamp and the working potential, and they find that in very many cases the cube root of the candle-power is proportional to the working potential minus a constant. Now, this is equivalent to saying that the

* "On the most Economical Potential-Difference to Employ with Incandescent Lamps:" 'The Electrician,' March 7, 1885, p. 348.

candle-power of the lamp is proportional to the cube of the potential measured above a certain point.

I have examined some records of measurements of Edison lamps to put this law to further test, and the results are given below for two 16-candle Edison lamps measured by myself, and one 8-candle lamp measured by Prof. Jamieson.

Edison 16-candle Lamp.—No. 1.

Candle-power= K .	Volts, v .	$\sqrt[3]{K}$.	$v-57.17$.	$\frac{v-57.17}{\sqrt[3]{K}}$.
16	105.27	2.5726	48.10	18.8
11.5	99.56	2.2572	42.39	18.8
8.25	94.32	2.0206	37.15	18.4
4.8	90.03	1.6869	32.86	19
2.8	85.74	1.4095	28.67	20
2	81.22	1.2599	24.05	19.08

Edison 16-candle Lamp.—No. 2.

K .	v .	$\sqrt[3]{K}$.	$v-50$.	$\frac{v-50}{\sqrt[3]{K}}$.
16	103.84	2.5198	53.84	2.5
13	100.03	2.3513	50	21.3
9.5	97.65	2.118	47.65	22.4
8.25	94.79	2.019	44.79	22.1
7	92.89	1.913	42.89	22.3
5.5	88.12	1.765	38.12	22.2

Edison 8-candle Lamp used by Professor Jamieson.

K .	v .	$v-28.7$.	$\sqrt[3]{K}$.	$\frac{v-28.7}{\sqrt[3]{K}}$.
5.2	45.9	17.2	1.7324	9.928
6.2	46.7	18	1.8371	9.798
8.2	48.3	19.6	2.0165	9.720
12.9	51.9	23.2	2.3453	9.892
14.3	53	24.3	2.446	9.937
21.3	56.2	27.5	2.772	9.921
25.3	58	29.3	2.936	9.981
35.8	61.1	32.4	3.296	9.831
43.8	63.1	34.4	3.525	9.759

In these cases we see that the cube root of the candle-power is very nearly proportional to the excess of the electromotive force above a certain point. Now, on examining a number of such cases, it appears that this constant is the value of that

electromotive force at which the lamp just begins to give signs of incandescence. In the case of the 8-candle lamp above, Professor Jamieson marks in his table against the E.M.F. 28.7 "filament bright red." It will be very interesting if further examination should give confirmation to this surmise; but the above figures seem to indicate that the cube root of the candle-power is proportional to the electromotive force reckoned from the neighbourhood of that pressure at which the filament begins to give out light. If we call this excess pressure the "effective volts," then we can state the rule that the cube root of the candle-power is proportional to the effective volts. We have then, according to the observations of Professors Ayrton and Perry, in a certain number of cases an empirical law of this kind,

$$\sqrt[3]{K} = a(v-b),$$

in which K is the candle-power measured at a pressure v .

This may be written

$$\frac{1}{3} \log K = \log a + \log (v-b).$$

By ordinary algebraic theory we have

$$\log a = \overline{a-1} - \frac{1}{2}\overline{a-1}^2 + \frac{1}{3}\overline{a-1}^3 - \&c.;$$

$\therefore$ if $c=b+1$, we can write $\log (v-b)$ as equal to the series

$$\overline{v-c} - \frac{1}{2}\overline{v-c}^2 + \frac{1}{3}\overline{v-c}^3 + \&c.,$$

and

$$\log K = A + B\overline{v-c} + C\overline{v-c}^2 + D\overline{v-c}^3 + \&c.$$

If the supposition above made is correct, that b is a value not far from that pressure at which the lamp becomes incandescent, then k is seen to be an exponential function of the effective volts of a kind similar to that which has been found to reconcile very well the values of observed life and corresponding working electromotive force.

Other observers have before now called attention to the fact, that within a certain range the candle-power of a lamp varies approximately as the sixth power of the current passing*.

In the case of the above-mentioned Edison 16-candle lamp,

* At the British-Association Meeting at Montreal, Mr. Preece read a note on this subject, confirming previous observations made in 1888, and showing, from observations of his own, Professor Kittler, and Captain Abney, that incandescence varies very nearly as sixth power of current.

No. 1, the following values were obtained, connecting current and candle-power :—

Candle-power, K.	$\sqrt[6]{K}$.	Current in amperes, <i>a</i> .	$\sqrt[6]{K \div a}$.
16	1.5874	.7582	2.09
11.5	1.5024	.7065	2.12
8.25	1.4215	.6720	2.11
4.8	1.2988	.6203	2.09
2.8	1.1872	.5859	2.03
2	1.1225	.5514	2.02

The table shows that, with considerable accuracy, over a range from two to sixteen candles, the incandescence varies as the sixth power of the current.

In those characteristics into which candle-power enters in any way, there is a considerable difficulty in getting results with sufficient accuracy to determine the constants of the characteristic equations. There is, however, one pair of variables, namely electromotive force and resistance, both of which can be measured with very high accuracy over a great range; and some very interesting examples of these pressure-resistance curves are given by Professor Jamieson in his memoir above alluded to.

A very short examination of the way in which an incandescence lamp behaves under increasing electromotive force, shows that the resistance decreases with increase of electromotive force, but that it does not decrease without limit; it tends to a minimum value, beyond which it appears to be constant. This is very strikingly shown for some of the Swan lamps tested by Professor Jamieson. In his paper certain formulæ are given connecting various lamp-constants, and as a first approximation to a pressure-resistance equation is given the following :— $\log P = \log E + ar$,

where *P* is a constant, *E* = E.M.F., and *r* = resistance.

It is not possible that such a formula should represent correctly the relation of resistance to pressure at high pressures, because it in no way expresses the fact that resistance tends to a minimum with increasing electromotive force.

An empirical formula can, however, be obtained which will express this in the following way :—

Let R be the resistance of a lamp measured with any electromotive force, E , at the terminals. Let E_0 represent the electromotive force at which the lamp just becomes incandescent, and let R_0 be the corresponding resistance. Let r be the minimum resistance to which the lamp approximates, as E increases, until the filament breaks. Then

$$R = r + 10^{\log(R_0 - r) - A(E - E_0)}$$

is an expression which has the following properties:—

If $E = E_0$, then $R = R_0$;

if $E = \infty$, „ $R = r$.

The above formula is otherwise written

$$\log(R - r) = \log(R_0 - r) - A(E - E_0).$$

This formula has been tested for an Edison 8-candle lamp of which the resistance was measured over a considerable range by Professor Jamieson, and the resistance-pressure curve given in his paper. For this lamp luminosity commenced at 28·7 volts, and at this pressure it had a resistance of 73·4 ohms; its resistance gradually decreased with increasing pressure until it became apparently constant and equal to 53·5 ohms. Taking A equal to $\frac{1}{50}$, we have

$$50 \log(R - 53\cdot5) + (E - 28\cdot7) = 50 \log(73\cdot4 - 53\cdot5) = 64\cdot945.$$

Calculating R by this formula for various values of electromotive force E , we have the following table of observed and calculated resistances:—

Edison 8-candle Lamp.

E.M.F.	Resistance observed.	Resistance calculated by above formula.	E.M.F.	Resistance observed.	Resistance calculated by above formula.
28·7	73·4	73·4	48·3	62·7	61·57
32·7	71·1	70·05	51·9	61	60·34
36·1	68·1	67·65	53	60·6	60
39·7	66·4	65·49	56·2	59·3	59·13
43·1	64·7	63·78	58	58·4	58·66
45·9	63·7	62·51	61·1	57·8	57·98
48·7	63·3	62·19	63·1	57	57·58

The accordance between the observed and calculated resistances is fairly close. It seems very probable, from an inspection

of the values at very low electromotive forces, that the resistance is a function of the electromotive force, reckoned from the pressure b , at which the lamp-filament begins to be bright red, and that of R is the resistance corresponding to any electromotive force, E , and r is the minimum resistance to which the lamp tends. Then

$$\log (R-r) = A + B(E-b) + C(E-b)^2 + \&c.,$$

where A B C , &c. are constants. If this should be the case, then it may be possible to express the life, candles per horse-power, candle-power, and resistance of a lamp, all as similar functions of the electromotive force, knowing certain constants and the two values of electromotive force at which the lamp becomes incandescent, and the resistance to which it finally tends to obtain. When, however, we are dealing with comparatively small variations of electromotive force, it is possible to calculate both life and efficiency from a simpler exponential function of the form—

$$l \propto v^x$$

$$R \propto v^y$$

when x and y are numerics. Assuming such an approximate formula, it becomes possible to deal with an interesting question, which has been discussed also by Professors Ayrton and Perry*, but which is capable of being investigated in a slightly different manner from that adopted by them in their paper. We shall take the warrant we have in the above figures for the assumption that for electromotive forces not far from those at which the lamp is intended to be used in the case of an Edison lamp, say 100 volts,

Average life varies inversely as the twenty-fifth power of the electromotive force ;

Efficiency, or candles per horse-power, varies as the fourth power of the electromotive force ;

Candle's light varies as the sixth power of the current, and therefore as the sixth power of the electromotive force, seeing that the resistance alters very little after the lamp has reached fair incandescence ;

And that therefore life varies inversely as the sixth-and-a-quarter power of the efficiency, and also inversely as the fourth-and-a-quarter power of the candle-power.

* "On Potential-difference, &c."

Let p be the price of a lamp in pounds sterling, or fractions of a pound.

Let l be the average life in hours, c the actual candle-power, and k the candles per horse-power, when run at a certain electromotive force v .

Then $\frac{p}{cl}$ is the cost of one candle-light per hour as far as the lamp itself is concerned. Let P be the cost of 1 horse-power hour of electric energy expended in the filament. P will not be the same for all amounts, 1000 P costs less than ten times 100 P , but for the small variations we are considering we shall consider P to be a constant. Then $\frac{P}{k}$ is the cost of the power in making one candle-light for one hour, and the total cost of getting one candle-light for one hour is

$$\frac{p}{cl} + \frac{P}{k} = T.$$

The first term is the expense of the translating device, and it may be called the cost of *lampage*.

The second term is the cost of the production of the energy which passes through the translating device or lamp-filament and is converted into eye-affecting radiant energy, and this may be called the *machinage*; hence the total expense of keeping going incandescent light is made up of lampage and machinage. Now, we can procure a given amount of light either by running the lamps very high, in which case lamps will cost a great deal, and power less in proportion, or we can run the lamps very low and save in lampage, whilst expending more in power in a greater number of lamps; and the question arises, apart from capital expenditure, at what point is the greatest economy obtained, or, in other words, what proportion ought lampage to bear to machinage in order that the total cost may be a minimum? To solve this we shall assume, as is very probable, that for the limit of variation of E.M.F. employed the average life of the lamp is an exponential function either of the candles per horse-power k , or of the candle's light c , for the particular lamps considered.

Let $l = \frac{A}{k^\alpha}$ and $l = \frac{B}{c^\beta}$ be the functions. α and β have definite values at a given E.M.F.; A and B are constants.

Then $cl = B'l^{\frac{1-\beta}{\beta}}; \quad k = A'l^{-\frac{1}{\alpha}}.$

Substituting, we have $B'pl^{\frac{1-\beta}{\beta}-1} + A'Pl^{\frac{1}{\alpha}-1} = T;$

in which B' and A' are constants.

Now this expresses the total cost of working as a function of the average life at a certain E.M.F.

Let us also take that $l = \frac{1}{v^\gamma}$, where v is the E.M.F. at which the lamps are being run.

Then $B'p\left(\frac{1}{v^\gamma}\right)^{\frac{1-\beta}{\beta}} + A'P\left(\frac{1}{v^\gamma}\right)^{\frac{1}{\alpha}} = T,$

or

$$B'pv^{-\frac{1-\beta}{\beta}\gamma} + A'Pv^{-\frac{\gamma}{\alpha}} = T.$$

Let this be varied by varying v ; to find at which value it becomes a minimum with respect to v , differentiate with respect to v , and equate to zero.

$$\frac{dT}{dv} = 0 = -\frac{B'\frac{1-\beta}{\beta}\gamma pv^{-\frac{1-\beta}{\beta}\gamma}}{v} - \frac{A'\frac{\gamma}{\alpha}Pv^{-\frac{\gamma}{\alpha}}}{v},$$

or

$$-\alpha\frac{1-\beta}{\beta}B'pv^{-\frac{1-\beta}{\beta}\gamma} = A'Pv^{-\frac{\gamma}{\alpha}}.$$

Hence the total cost of working is a minimum when the cost of power $A'Pv^{\frac{\gamma}{\alpha}}$ is to the lampage $B'pv^{-\frac{1-\beta}{\beta}\gamma}$ as $\alpha\frac{1+\beta}{\beta}$ to unity.

Now the above investigations show that for Edison lamps α is a quantity in the neighbourhood of $6\frac{1}{4}$, and β near $4\frac{1}{5}$; hence

$$-\alpha\frac{1-\beta}{\beta} = \frac{19}{4},$$

and hence ratio of lampage to total cost is $\frac{4}{23} = 17.4$ per cent.

Hence we arrive at this curious result, that independently of the cost of the lamp, or electrical energy, we must run at such a pressure that lampage is about 18 per cent. of the total cost. Now, if instead of employing these approximate exponential expressions for the values of l , c , and k in terms of the electromotive force, we had introduced the more accurate forms of equation indicated above, we should have had an

equation in terms of v to solve as the result of equating the differential to zero, which would, by the introduction of the proper constants, give the value of electromotive force at which the total cost becomes a minimum. In their paper, Professors Ayrton and Perry have calculated to a fraction of a volt what this economical potential is. As, however, the characteristic equations are only approximate, it seems hardly necessary to do more than obtain a similar approximate expression for the economical working.

At the Edison lamp-factory in America calculations were made, on the assumption of a particular type of lamp and length of life and cost of power, to ascertain the ratio of lampage to total cost, which made the total cost a minimum, and the result appeared to be to fix it at about 16 per cent. These calculations are therefore in singular accord with the deduction of theory based on determination of the constants of the characteristic curves, arrived at both by graphic and analytical methods.

X. *On the Structure of Mechanical Models illustrating some Properties of the Æther.* By Prof. GEORGE FRANCIS FITZGERALD, F.R.S.*

THE elements of which the model is constructed consist of pairs of wheels so geared together that when one of them rotates it causes the second to rotate in the same direction. The simplest way of effecting this is to connect them by a band, and this is sufficient for a one-dimensional model. Such a model may be constructed by fixing a number of wheels with their axes parallel and at right angles to a plane, and connecting each wheel with its neighbours by elastic bands. This represents a nonconducting region of the æther. A perfectly conducting region is one in which there are no bands, and a partially conducting region would be represented by the bands slipping more or less. A short description of how electrostatic, electrokinetic, and luminiferous phenomena are illustrated by such a one-dimensional model will be clearer than the corre-

* Read March 28, 1885.

sponding description of the tridimensional model, the structure of which is the special purpose of this paper.

As an illustration of an electrostatic phenomenon, consider two conducting regions separated by a nonconducting region everywhere except along one line where the bands are removed. If anywhere in this line a rotation in opposite directions be communicated to the wheels that abut on it, then all the wheels in the nonconducting region will be turned more or less. If anywhere two neighbouring wheels turn equally, there is no straining of the band connecting them; but if one turn more than the other, the connecting band is strained and one side becomes tight and the other loose. Now it will be found in the case considered that all the bands are strained, and that all the tight sides are turned towards one conductor and all the loose sides towards the other. This represents the charging of the two conductors in opposite ways. The strain of the bands in any element of the medium represents the polarization of the element, and the line joining the tight and loose sides is the direction of polarization or of electric displacement. The energy of the system is in the form of this straining of the bands, which produces stresses tending to restore the unstrained condition. With a given strain at their respective surfaces, there would be more elements involved and more energy in the medium when the conductors are far apart than when near, showing that if we could represent in any way the fact that conductors can move through the æther there would be forces tending to produce this motion, or, in other words, there would be attraction between these oppositely electrified bodies. As, however, the model does not illustrate the connection between matter and æther, neither this nor magnetic attractions are represented, nor have electromotive forces such as exist in cells been represented. If the forces that have been supposed to turn the wheels along the conducting line connecting the two conducting regions cease to act, the state of strain will disappear, and what represents an electric discharge along this line will take place. All along this line, during the discharge, the wheels at opposite sides will be rotating in opposite directions; so that this is what represents an electric current at any point. It is the same as an electric displacement; and in a nonconductor such opposite rotation is resisted by the stresses

in the band, but in a conductor it may take place to any extent. During the discharge the whole of the nonconducting region is full of rotating wheels, and their axes of rotation are at right angles to the direction of discharge. Their velocity of rotation evidently represents the magnetic force accompanying the discharge, and the momentum of the wheels represents the electrokinetic momentum of the current, *i.e.* its self-induction. This is further illustrated by this, that if the frictional resistance be small enough, this momentum will carry the wheels beyond their positions of equilibrium, and there will result an oscillating discharge, such as occurs when an electric condenser is discharged through a sufficiently small resistance. If we suppose a certain amount of frictional resistance at any point along the line of discharge, we may see that the energy expended on friction is conveyed to the place by the bands in the surrounding nonconductor and comes in at the side of the conductor, in accordance with Prof. Poynting's theorem as to the direction of the flow of energy in an electrodynamic system.

The mutual induction of two circuits may also be illustrated by the model. Sufficient has been explained, however, to show how electrostatic and electrokinetic phenomena are represented on the model. If a sudden movement of rotation be communicated to any set of wheels, it is evident that inertia will prevent their neighbours being instantaneously turned, while the connecting bands will be strained. Rotation will, however, be communicated to the neighbouring wheels, and from them to their neighbours, by a process which is a species of wave-propagation. If we consider the nature of the disturbance which is thus propagated, we see that it consists in a rotation whose axis is at right angles to the direction of propagation, and of a polarization of the bands which is at right angles both to the axis of rotation and to the direction of wave-propagation. These are respectively a magnetic and an electric displacement, which are at right angles to one another and to the direction of wave-propagation. This is exactly in accordance with Maxwell's electromagnetic theory of light-propagation. It is thus seen how the same model that can represent electrostatic and electromagnetic phenomena also illustrates luminiferous phenomena by its small oscillations.

If we try to produce a tridimensional model by means of wheels geared together by bands, we are met by the following difficulty. The energy of the model we have been considering may be represented in the following way:—Let ξ represent the angular rotation of any wheel from a given position; then the kinetic energy of an element will evidently be proportional to ξ^2 , while the potential energy of an element, depending as it does on the difference of rotation of neighbouring wheels, will be proportional to $\left(\frac{d\xi}{dx}\right)^2 + \left(\frac{d\xi}{dy}\right)^2$. If we were to build up a tridimensional model by simply putting together three such systems of wheels in three rectangular planes, the kinetic energy of an element of the model would be proportional to

$$\xi^2 + \eta^2 + \zeta^2,$$

but its potential energy would be simply

$$\frac{d\xi^2}{dx^2} + \frac{d\xi^2}{dy^2} + \frac{d\eta^2}{dz^2} + \frac{d\eta^2}{dx^2} + \frac{d\xi^2}{dy^2} + \frac{d\xi^2}{dz^2},$$

instead of

$$\left(\frac{d\xi}{dy} - \frac{d\eta}{dz}\right)^2 + \left(\frac{d\xi}{dz} - \frac{d\eta}{dx}\right)^2 + \left(\frac{d\eta}{dx} - \frac{d\xi}{dy}\right)^2,$$

which is what it should be if it is to represent the æther.

The electrostatic and electrokinetic energies of an element of the æther may be expressed in this form by assuming ξ , η , and ζ such that their velocities $\dot{\xi}$, $\dot{\eta}$, and $\dot{\zeta}$ are α , β , and γ , the components of the magnetic force at the element. Now simple band-gearing will not enable us to arrange that no straining shall result, when, for instance, $\frac{d\eta}{dx} = \frac{d\xi}{dy}$, but this

may be accomplished by the following arrangement:—Each element of the æther is to be represented by a cube, on each edge of which there is a paddle-wheel. Thus on any face of the cube there will be four paddle-wheels. Now, if any opposite pair of paddle-wheels on a face rotate by different amounts, they will tend to pump any liquid in which the whole element is immersed into or out of the cube, and if the sides of the cube be elastic there will be a stress which will tend to stop this differential rotation of the wheels. If, however, the other pair also rotate by different amounts they may undo what the

first pair do ; and thus the stress will depend on the difference between the differential rotations of these opposite pairs of wheels, i. e. on $\frac{d\eta}{dx} - \frac{d\xi}{dy}$.

In order that these four wheels may not similarly work with any other wheel in the cube it is necessary to place diaphragms, cutting the cube into six cells, each a pyramid standing on a face of the cube. These must be so made that liquid may not be able to pass from one cell to another through the diaphragm, nor beside the paddle-wheels. In order actually to effect this, the floats on the paddle-wheels would have to be drawn down while passing the diaphragm (of course these mechanical details could hardly be carried out so as to work with sufficiently little friction for the working of any actual model to approximate sufficiently closely to that of the æther for it to be worth while attempting to construct it). The faces of the cube should be filled up with diaphragms past which the paddle-wheels should pump liquid, and whose elasticity should be the means of storing electrostatic energy in the medium. It may be worth while pointing out some of the results of having shown that the energy of a disturbance of this medium would be represented by the same equations as those Maxwell has shown to hold for the æther. The most complicated results follow from supposing the faces of the cubes, of which the medium is constructed, to have different elasticities. Such a structure represents a crystalline medium. Its vibrations would be propagated according to the laws of propagation of light in crystalline media. The wave-surface would be Fresnel's wave-surface, and it would exhibit conical refraction. If the cubes were twisted, the structure would be like that of quartz or other substances that rotate the plane of polarization of light. In order to represent the rotation of the plane of polarization of light by magnetism, it would be necessary to introduce some mechanism connecting this æther with matter. That this would be required is evident from the consideration that no amount of inherent rotation of these wheels would alter the plane of polarization of a vibration transmitted by them. Now, although I am not prepared to suggest any actual method of connection, it may be worth while pointing out two different ways in which it may work.

In the first place the matter may be in rotation when subject to magnetic force ; and it may be connected with the æther in such a way that the direction of its axis and rotation is altered by the vibrations passing in the æther. It would then react on the æther in such a way as to rotate the plane of polarization of the wave. In the second place the matter might only act as a link connecting æther elements rotating in rectangular directions in such a way that the rotation of one element altered the axis of rotation of a rectangular element. The reaction of this latter on the former would then rotate the plane of polarization of a wave propagated parallel to the latter. The first of these connections is that in accordance with Maxwell's theory as to the connection between the æther and matter that he introduces in his 'Electricity and Magnetism,' in order to explain the rotation of the plane of polarization of light by magnetism ; while the second is the one that I supposed in my paper "On the Electromagnetic Theory of the Reflection and Refraction of Light" (R. S. Trans. 1880).

I need hardly say, in conclusion, that I do not in the least intend to convey the impression that the actual structure of the æther is a bit like what I have described. What physicists ought to look for is such a mode of motion in space as will confer upon it the properties required in order that it may exhibit electromagnetic phenomena. Such a mode of motion would be a real explanation of these phenomena. I have only given a description of them.

I think, however, that it is worth while considering these models, because in them the disturbance which represents light is not the same as the vibrations of an elastic jelly, for what represents an electric displacement is a change of structure of an element, and not a displacement of the element ; and it seems almost certain that, notwithstanding the very high authority which seems to support the view that the æther is *like* an elastic jelly, nevertheless its vibrations are much more of the nature of alterations in structure than of displacements.

XI. *On Binocular Glasses adjustable to Eyes having unequal Focal Lengths.* By Colonel MALCOLM, R.E., C.B.*

WHETHER many people are born with their eyes differing in focal length or not I cannot say; it is sufficient that I have arrived at a period of life when glasses are necessary to enable me to read with comfort, and also that I found a very highly treasured pair of binocular glasses, by Voigtländer, become yearly more difficult to use, and at last *useless*; but I found that I could see with equal distinctness with each eye, using only one eye at a time, on condition of a very slight alteration of focus.

Mr. Browning, optician, in the Strand, having tested my eyes for a pair of pince-nez, told me that, as regards the ordinary reading and writing distances at any rate, my eyes were not a pair, and that after a certain age few people's were who had used their eyes much.

Accepting the fact, I set to work to make my binocular glasses once more useful—in this way.

One tube is left untouched; the eyepiece of the other is so arranged that it can be moved through a small range in and out, with reference to the eyepiece of the untouched tube, by turning round a milled ring. An index arrangement is provided.

The unaltered tube is used with one eye and brought to the most perfect focus possible in the ordinary way; then the other tube is used with the other eye, and by means of the adjustment its definition is made as perfect as may be, the ordinary adjustment not being interfered with. The two eyes are then used together; and the process of adjustment had better be gone over again, as certainly the two eyes do help each other.

The final position of the index-mark is noted; and that holds good for all ranges, as far as I have tried.

Having noted this, you may lend your glasses to your friend, who may alter them to his sight, and yet have them in perfect order for yourself by bringing the index to your own mark.

* Read March 28, 1885. The glasses were exhibited at the Meeting.

The increase of power in the glasses, and of comfort to the user of them, has been a surprise to almost all who have tried the improved adjustment.

I do not enlarge upon the way in which Mr. Browning has carried out the details; you can see for yourselves that the improvement in no way disfigures the glasses, and as the expense is 12s. 6d. for glasses made to the common pattern, I think that the improvement is within the reach of most people who use binoculars.

XII. *Summary of Lecture on Calculating Machines, delivered before the Physical Society of London, March 28, 1885, by JOSEPH EDMONDSON, of Halifax.*

CALCULATING machines are of two classes, the automatic and the semi-automatic. The former were invented by Mr. Charles Babbage between 1820 and 1834, and were designed mainly for the computation of tables. When the primary factors of a table are placed on such a machine, it will calculate the table and impress it line by line upon a stereo-mould with great celerity and absolute correctness. Mr. Babbage's machine, so far as it was completed, is in the South Kensington Museum; and another machine by M. Scheutz (for the principles of which he confessed his indebtedness to Mr. Babbage) is in the office of the Registrar General at Somerset House. These machines proceed by the method of Differences, and are therefore termed "Difference-Engines." Their work is limited to addition, but they indirectly perform subtraction by adding the complement of the subtrahend.

If it be required to construct a table (of cubes for instance), the law of its growth must be found by reckoning mentally the first few terms of the table. Placing them in a line, the difference between each term and its successor is placed under the former. This process is repeated with the differences until at last a line is reached in which all the differences are cyphers. Thus:—

Table . . .	A	1	8	27	64	125	216
1st difference	B	7	19	37	61	91	
2nd "	C	12	18	24	30		
3rd "	D	6	6	6			
4th "	E	0	0				

The machine may now be set to work by placing the first column of figures in their respective places upon it. During the first half of the first revolution of the machine, the wheels containing the terms A and C have the terms B and D added on to them and become 8 and 18 respectively. During the second half-revolution, B and D have the terms C and E added on to them, becoming 19 and 6 respectively, and the second column is complete. Each succeeding revolution of the machine similarly produces each succeeding column, the upper line being the term required for the table.

The apparatus consists of identical parts, equal in number to the digits required for the last column of the tables it is to compute. It is therefore very complex and very costly, and will not serve the purpose of computers in general, who must have recourse to the semi-automatic class of instruments. These are portable, of moderate cost, allow of very rapid working, and require no special mathematical skill.

In 1663 Sir Samuel Moreland produced an instrument by which additions and subtractions could be worked, digit by digit; but it took more time than the ordinary mental operation. It was left to Viscount Mahon (afterwards Earl of Stanhope) to produce the first really practical instrument. Besides a machine dated 1780, which was a great advance on that of Sir Samuel Moreland, though on the same lines, he invented three machines. Those of 1775 and 1777 were on the table for inspection after the lecture, and abound in beautiful and effective contrivances. The third machine the lecturer had not seen, and it has never been described. In that of 1775 is found the "Stepped Reckoner," the basis of the only instruments that have come into extensive use.

The reckoner of the modern machines, patented in 1851 by M. Thomas de Colmar, consists of a cylinder divided into 10 sections, on which there are respectively 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9 teeth. The teeth of one section being coincident with an equal number of those of the next section, the whole presents a stepped appearance. Each revolution of this reckoner moves a pinion of 10 teeth, in gear with one of its sections, as many teeth as there are upon that section. The motion of the pinion is communicated to a dial with the digits 0 to 9 in orderly succession upon it. Thus, if the figure seen

through an aperture in the covering of the dial were 0, and the pinion were in gear with the section having 3 teeth, the first revolution of the reckoner would move the dial to 3; the second to 6; the third to 9; this result being the multiple of 3 (teeth on the reckoner) by 3 (revolutions). A series of, say, 8 reckoners, pinions and dials, each pinion being set to the section having 3 teeth, would give in 3 revolutions the product 99,999,999. The next revolution would bring in a new feature—the carrying of the tens. Here lies the great difficulty of Calculating Machines; but the difficulty has been overcome, though not without leaving room for improvement. The 1 that will have to be carried from each dial to that immediately on its left cannot be added while the latter is being operated upon by its reckoner. The machine must therefore make an independent note of it until it can be dealt with. Each dial has a tooth (the *primary* carrying-tooth) which makes this note at the moment that the dial passes from 9 to 0, by pushing along the axis of the reckoner belonging to the next higher digit a sliding-piece which revolves with the reckoner and carries a single tooth (the *secondary* carrying-tooth). Above this sliding-piece, and upon the same axis as the pinion in gear with the reckoner, is a second pinion of 10 teeth. Before the sliding-piece was moved by the primary carrying-tooth, the secondary carrying-tooth passed on the side of this second pinion without affecting it; but being now brought into line with it, the pinion will be moved 1 tooth, and will therefore add 1, as soon as the reckoner has added its own proper number; so that the fourth revolution, giving the product of 33,333,333 by 4, brings the dials to 133,333,332. When the secondary tooth has done its work, an incline on its side comes in contact with a projection on the frame of the machine, which restores it to its original inoperative position. Each reckoner is set to operate a little later than its neighbour to the right, so that the carrying on the latter may be completed before that on the former is commenced. The before-named blank space on the reckoner is to allow time for the carrying; and for a “dead-point” in the machine, at which the first-named pinions, which are movable longitudinally on their axes, may be set to their required sections of the reckoners, and other changes may be made in the setting of the

machine. The position of the pinions is shown by indicators on the face of the machine, each of which can be set to any required figure.

Each pinion communicates its motion to its dial by bevel-gear in the well-known combination of three wheels, which permit the action to be reversed at will, so that the change from $+$ to $-$ or from $\times$ to $\div$ is made instantaneously.

In multiplying by more than a single digit, the dials require to change their position in relation to the multiplicand, to imitate the "stepping" of the lines in long multiplication. They are therefore placed in a slide which can be moved, step by step, from right to left or *vice versa*.

Up to a recent date the machine of M. Thomas de Colmar was the only one in use. A few years ago Mr. George B. Grant, of Boston, Mass., brought out an instrument somewhat on the lines of the Stanhope machine of 1777. It is very compact, is beautifully and substantially made; but as subtraction is performed by adding the complement of the subtrahend, reversing is so tedious as to be fatal to the general adoption of the machine. Its range is also very limited, and there are other defects inherent in the method of its construction.

Still more recently Mr. Tate, of London, has introduced a machine on the lines of M. Thomas, in the internal mechanism of which there are several improvements. The workmanship, too, is very superior to that of the French instrument.

Mr. Edmondson (the lecturer) having, in practice, found certain inconvenient limits to the utility of these machines, conceived the idea of obviating them by throwing the instrument into the circular form, and thus obtaining an *endless* instead of a limited slide. This has produced a machine of greatly extended powers; for it can deal with a multiplier or dividend of any number of figures, and can carry a quotient to any number of decimal places. But its chief utility does not consist in dealing with such large numbers. It frequently happens that quotients produced on a machine require to be further operated upon; but as in the straight machines they are recorded on a special set of dials, which are not in the general working line, they must be transferred by hand before they can be dealt with in any other way than as multipliers.

In the circular machine they come up in the working line ready for further operations. The circular form, in fact, gives a certain elasticity in the applications of the machine which could not be described within the limits of the lecture, and which can only be appreciated by actual use of the instrument. Although the special set of dials for recording multipliers and quotients are dispensed with as non-essential in the circular machine, they could be easily and conveniently added, and would still further extend the powers of the instrument.

The lecturer expressed his indebtedness to the following gentlemen for the loan of valuable instruments on the table, which, along with Mr. Tate's machine and several belonging to the lecturer himself, attracted much attention after the lecture :—

General Babbage, Bromley, Kent. A piece of the Difference-Engine of the late Charles Babbage, Esq.; Stanhope Machines of 1775 and 1777; Sir Samuel Moreland's Machine of 1663.

The Rev. Professor Harley, F.R.S., Huddersfield. Stanhope Machine of 1780.

Theodore B. Jones, Esq., Harrogate. The American Machine of George B. Grant, Esq.

F. H. Bowman, Esq., D.Sc., Halifax. A Machine of Sir Samuel Moreland's, constructed on the principle of "Napier's Bones."

XIII. *On a Chrono-barometer and Chrono-thermometer.*

By W. FORD STANLEY.*

THESE instruments consisted of clocks regulated by pendulums, formed, in the first instrument, of a mercurial barometer, and in the second of a similar barometer inclosed in a hermetically-sealed air-chamber, the inclosed barometer thus acting as an air-thermometer. Increase of pressure in the one case, and of temperature in the other, causes the mercury to rise, and thus accelerates the pendulum. By the gain or loss of time the mean pressure or temperature can be calculated for any period.

* Read March 28, 1885.

OF A PAPER MACHINE AND OTHER THINGS
The paper machine is a very important
part of the paper mill. It is the
machine which takes the pulp and
forms it into sheets. The sheets are
then dried and finished. The machine
is very complicated and has many
parts. It is usually made of iron
and steel. The machine is used to
make paper of various kinds. It is
used to make paper for books, newspapers,
and other things. The machine is
very important in the paper mill.
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PROCEEDINGS
OF
THE PHYSICAL SOCIETY
OF LONDON.

AUGUST 1885.

XIV. *On the Theory of Illumination in a Fog.*

By Lord RAYLEIGH, F.R.S.*

As a step towards a better understanding of the action of fog upon light, it seems desirable to investigate what the phenomena would be in the simplest case that can be proposed. For this purpose we may consider the atmosphere and the material composing the fog to be *absolutely transparent*, and also make abstraction from the influence of obstacles, among which must be included the ground itself.

* Conceive a small source of radiation, *e. g.* an incandescent carbon filament, to be surrounded by a spherical cloud, of uniform density, or at any rate symmetrically disposed round the source, outside of which the atmosphere is clear. Since by hypothesis there is no absorption, whatever radiation is emitted by the source passes outward through the external surface of the cloud. The effect of the cloud is to cause diffusion, *i. e.* to spread the rays passing through any small area of the surface (which in the absence of the cloud would be limited to a small solid angle) more or less uniformly over the complete hemisphere.

Whether the total radiation passing outwards through the small area on the external surface of the cloud is affected by

* Read April 25, 1885.

the existence of the cloud depends upon the circumstances of the case. If it be laid down that the total emission of energy from the source is given, then the presence of the cloud makes no difference in respect of the energy passing any element of the spherical area. But this supposition does not correspond to a constant temperature of the source, in consequence of the energy received back from the cloud by reflection. To keep the total emission of energy constant, we should have to suppose a rise of temperature increasing indefinitely with the size and density of the cloud.

Let us now suppose that the region under consideration is bounded upon all sides by a distant envelope of perfect reflecting-power. Then, whatever the density of the clouds which may wholly or partially occupy the enclosure, we know, by the second law of thermodynamics, that at every internal point there is radiation in every direction of the full amount corresponding to the temperature of the source. In one sense this conclusion holds good, even although the matter composing the cloud has the power of absorption. But in that case equilibrium would not be attained until the clouds themselves to the remotest parts had acquired the temperature of the source; whereas under the supposition of perfect transparency the temperature of the cloud is a matter of indifference; and equilibrium is attained in a time dependent upon that required by light to traverse the enclosure. So far we have made no supposition as to the distribution of the cloud; but we will now imagine a layer of such thickness as to allow only a very small fraction of the incident radiation to penetrate it, to line the interior of the reflecting envelope. This layer itself plays the part of a practically perfect reflector; and it is not difficult to see that the reflecting envelope hitherto conceived to lie beyond it may be removed without interfering with the state of things on the inner side of the layer of cloud. We thus arrive at the rather startling conclusion that at any distance from the source, and whatever the distribution of clouds, there is always in every direction the full radiation due to the temperature of the source, provided only that there lie outside a complete shell of cloud sufficiently thick to be impervious. And this state of things is maintained without (on the whole) emission of energy from the source.

Even if the material composing the cloud possesses absorbing-power for some kinds of radiation, *e. g.* for dark radiation, but is perfectly transparent to other kinds, *e. g.* luminous radiation, the general theorem holds good as respects the latter kinds; so that in the case supposed the light would still be everywhere the same as in a clear enclosure whose walls have throughout the same luminosity as the source. But in order to compensate the absorption of dark rays, the source must now be supplied with energy.

Some of the principles here enunciated have an acoustical as well as an optical application, and indeed first occurred to me some years ago in connection with Prof. Tyndall's investigations upon fog-signals. The effect of "acoustic clouds" analogous to fog (and unattended with absorption of energy), might be very different upon the report of a gun and upon the sustained sound of a siren, the latter being reinforced by reflection from the acoustic fog.

The theory presented in the present paper may be illustrated by the known solution of the comparatively simple problem of a pile of transparent plates*. If ρ denote the proportion of the incident light reflected at a single surface, then the proportion reflected $\phi(m)$, and transmitted $\psi(m)$, by a pile of m plates is given by

$$\frac{\phi(m)}{2m\rho} = \frac{\psi(m)}{1-\rho} = \frac{1}{1+(2m-1)\rho}.$$

From these expressions it is evident that, however small ρ may be, *i. e.* however feeble the reflection at a single surface, we have only to suppose m large enough in order that the reflection may be as complete, and the transmission as small, as we please. Such a pile may, under ordinary conditions, be regarded as impervious.

But now suppose that after passing the pile of m plates, the light is incident upon a second pile of n plates, and consider the intensity between the two piles, the original intensity being unity, as before. For the intensity of the light travelling in the original direction we have

$$\begin{aligned} & \psi(m) + \psi(m) \cdot \phi(n) \cdot \phi(m) + \psi(m) \cdot \{\phi(n) \cdot \phi(m)\}^2 \\ & + \psi(m) \cdot \{\phi(n) \cdot \phi(m)\}^3 + \dots; \end{aligned}$$

* Stokes, Proc. Roy. Soc. vol. xi. p. 545 (1862).

or on summation of the geometric series,

$$\frac{\psi(m)}{1 - \phi(n) \cdot \phi(m)}.$$

If we introduce the values of ϕ and ψ in terms of m, n, ρ , this becomes

$$\frac{2n\rho + 1 - \rho}{2(m+n)\rho + 1 - \rho}.$$

In like manner, for the light going the other way we have

$$\frac{\phi(m) \cdot \phi(n)}{1 - \phi(m) \cdot \phi(n)};$$

or in terms of m, n, ρ ,

$$\frac{2n\rho}{2(m+n)\rho + 1 - \rho}.$$

When m and n are great, both expressions reduce to $n/(m+n)$; so that the light passing in the two directions is equally bright. Moreover, and this is the point to be especially noticed, however great m may be—that is, however impervious the first pile is, the light between the two piles may be made to approach the original light in brightness as nearly as we please, by sufficiently increasing the number of plates in the second pile; that is, the light between the piles may be made to be the same as if the first pile were removed. From this example we may understand more clearly how a very small quantity of light penetrating directly may be beaten backwards and forwards, as between two reflectors, until the original intensity is recovered.

XV. *A Monochromatic Telescope, with Application to Photometry.* By LORD RAYLEIGH, F.R.S.*

THE purpose of this instrument is to exhibit external objects as they would be seen either with the naked eye, or through a telescope, if lighted with approximately monochromatic light; that is, to do more perfectly what is done roughly by a coloured glass.

The arrangement is not new, though I am not aware that it has ever been described. In 1870 I employed it for deter-

* Read April 25, 1885.

minations of absorption, and, if my memory serves me right, I heard soon afterwards from Clerk-Maxwell that he also had used it. It is, indeed, a very slight modification of Maxwell's colour-box.

In the ordinary form of that instrument, white light admitted through a slit is rendered parallel by a collimating lens, dispersed by flint-glass prisms, and then brought to a focus at a screen, upon which accordingly a pure spectrum is formed. This screen is perforated by a second slit, immediately behind which the observer places his eye. It is evident that the light passing the aperture is approximately monochromatic, so that the observer, if he focuses his eye suitably, will see the prism illuminated with this kind of light. The only addition now required to convert the instrument into a monochromatic telescope is a lens placed just within the first slit, of such power as to throw an image of external objects upon the prism or diaphragm upon which the eye is focused. If desired, an eye-lens may be placed at the second slit; but this is not generally needed.

In the present instrument a direct-vision dispersing prism is used, so that the optical parts can be all disposed in a narrow box of nearly 3 feet in length. The lenses are all single lenses, and work sufficiently well. The slits are of such width that either coincides with the image of the other, and their relative position is so chosen that the mean refrangibility of the light is that corresponding to sodium. Objects seen through the instrument thus appear as if lighted by a sodium flame.

The principal object which I had in view in the construction of the instrument now exhibited was to see whether it could be made of service in the comparison of compound lights of somewhat different colours—a problem just now attracting attention in connection with electric lighting. It is scarcely necessary to say that a comparison of this kind is physically incomplete unless it extends to all the spectral components separately; but for commercial purposes such an extended comparison is too complicated, and indeed useless. Determinations at two points of the spectrum, as proposed by Capt. Abney, would certainly suffice for ordinary purposes; and in view of the convenience of expressing the result by a

single number, it is not unlikely that people practically concerned in these matters will content themselves with a comparison at one point. It seems desirable that some convention should be arrived at without much further delay; so that two lights should be considered to be commercially equal, if they have the same intensity of, *e. g.*, sodium or of thallium light. It will be understood that such a mode of estimation assumes that the intensity varies along the spectrum in a gradual manner; and this consideration may tell against the use of the sodium light as a standard, inasmuch as the component in question often predominates unduly in candle-flames from the actual presence of sodium.

Whatever choice be made, an instrument like the present may be employed to make the desired selection; and it is applicable to any photometric arrangement. For my own experiments I have used the shadow-method, and find it possible to compare any bright sources, however different in colour. The only difficulty arises from the necessary enfeeblement of the light by selection, and this practically precludes observation with standard candles. With gas-flames and glow-lamps the light is sufficient.

XVI. *On the Self-Regulation of the Compound Dynamo.*

*By A. W. RÜCKER, M.A., F.R.S.**

THE theory of the self-regulating power of the compound dynamo has not been treated in a satisfactory manner in any discussion of the subject with which I am acquainted. Some conclusions as to the proper method of constructing self-regulating dynamos have been drawn from equations in which the power of perfect self-regulation is assumed. As this perfection is known to be unattainable, no great weight can be given to arguments based on the assumption in question, and it will in the following paper be abandoned. A further inquiry will also be entered upon, *viz.* how far the conditions of good self-regulation are consistent with the other desiderata aimed at in the construction of a dynamo—such, for instance, as high efficiency.

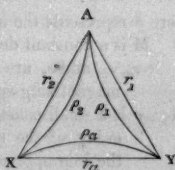
The entire investigation will be based on Frölich's equation,

Read April 25, 1885.

which expresses the electromotive force in terms of the current and the constants of the machine, and will therefore not apply in cases where the prejudicial effect of the magnetization of the armature is apparent. The notation and diagrammatic representation of the machine are those which I have found useful for teaching purposes.

Let the points A, X, Y be joined by three straight and three curved lines.

Let the side XY represent the armature of the machine. The points A, X, and Y may be called its terminals. The curved lines represent the magnetizing coils of the inducing magnets; the straight lines AX and AY represent external resistances.



Let the values of the resistances be as indicated in the figure. Let the total electromotive force be E , that between Y and A e_1 , and that between A and X e_2 . The currents in r_1 and ρ_1 may be represented by c_1 and γ_1 , and a similar notation used for the other conductors.

Finally, let the whole resistance be R , let the resistances of the multiple arcs between Y A and A X be R'_1 and R'_2 respectively, and let c' be the total current in each of these multiple arcs.

The current is to be conceived as flowing out of the armature at Y, passing on to the three conductors ρ_a , ρ_1 , and r_1 , and returning at X.

All the existing forms of dynamo can be represented by leaving out some of the conductors shown in the above symmetrical figure. Their properties are perhaps more easily recognized by regarding them as particular cases of a more highly generalized machine. The notation used has also another advantage. The shunt in the Compound Dynamo may be used in one of two ways—viz. as a shunt upon the armature alone, or as a shunt upon the armature and series-coils. It is usual to represent its resistance by a symbol which indicates only that it is a shunt, without showing in which of these ways it is used. If, however, it is agreed that r_2 shall always represent the external resistance, r_1 being in any prac-

tical case infinite, then ρ_1 always represents the series-coils, and ρ_a or ρ_2 the shunt-coil according as it is applied in the first or second of the methods above referred to.

If we adopt Frölich's equation we may write

$$E = \frac{nM(s_0\gamma_0 + s_a\gamma_a + s_1\gamma_1 + s_2\gamma_2)}{1 + \sigma(s_0\gamma_0 + s_a\gamma_a + s_1\gamma_1 + s_2\gamma_2)};$$

where n represents the number of revolutions;

M is a constant depending on the machine;

$s_a, s_1,$ and s_2 are the number of turns made by the magnetizing spirals $\rho_a, \rho_1,$ and ρ_2 ;

and σ is a small constant depending on the gradual weakening of the magnetizing effects of the currents as the iron approaches saturation.

The product $s_0\gamma_0$ represents the number of ampere-turns by which the initial magnetic field is, or may be supposed to be, produced. If γ_0 is an independent current, this quantity may be large. If it is only a fictitious current which is regarded as the cause of the permanent magnetism, $s_0\gamma_0$ is small. If the product of $s_0\gamma_0$ and of σ may be neglected, and if $M s_0\gamma_0 = k$, we may write

$$E = nk + \frac{nM(s_a\gamma_a + s_1\gamma_1 + s_2\gamma_2)}{1 + \sigma(s_a\gamma_a + s_1\gamma_1 + s_2\gamma_2)};$$

We have at once

$$\frac{\gamma_a}{\rho_a} = \frac{c'}{R'_1 + R'_2} = \frac{c_a}{\frac{1}{\rho_a} + \frac{1}{R'_1 + R'_2}},$$

$$\gamma_1\rho_1 = c'R'_1 \text{ and } \gamma_2\rho_2 = c'R'_2.$$

Hence

$$E - nk = \frac{nM \left\{ s_a \frac{R'_1 + R'_2}{\rho_a} + s_1 \frac{R'_1}{\rho_1} + s_2 \frac{R'_2}{\rho_2} \right\} c'}{1 + \sigma \left\{ s_a \frac{R'_1 + R'_2}{\rho_a} + s_1 \frac{R'_1}{\rho_1} + s_2 \frac{R'_2}{\rho_2} \right\} c'}.$$

But

$$c' = \frac{\rho_a c_a}{R'_1 + R'_2 + \rho_a} = \frac{\rho_a}{R'_1 + R'_2 + \rho_a} \times \frac{E}{R}.$$

Hence, if we write

$$\left\{ s_a \frac{R'_1 + R'_2}{\rho_a} + s_1 \frac{R'_1}{\rho_1} + s_2 \frac{R'_2}{\rho_2} \right\} \frac{\rho_a}{R'_1 + R'_2 + \rho_a} = S,$$

we get

$$E - nk = \frac{nMSE}{R} \left/ \left(1 + \frac{\sigma SE}{R} \right) \right.$$

or

$$(E - nk) \left(E + \frac{R}{\sigma S} \right) = \frac{nM}{\sigma} E.$$

If the machine has no external excitement k may be neglected, unless the conditions under which it begins to work are under investigation. Hence the equation becomes

$$E = \{nM - R/S\} / \sigma. \quad (1)$$

If there is external excitement and the magnets have not reached the condition where the evidences of approaching saturation become important, terms in σ may be neglected, and we have

$$E = \frac{nkR/S}{R/S - nM}. \quad (2)$$

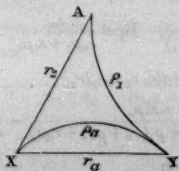
If k and σ be both neglected these equations lead to conclusions of no practical value; for if $nM = R/S$, both (1) and (2) make E indeterminate; and if $nM \neq R/S$, we get $E = \infty$ from (1) and $E = 0$ from (2).

These results prove that no true approximation to the behaviour of the machine can be obtained by neglecting both the constants k and σ .

The above equations are exactly similar in form to those ordinarily given as applying to series and shunt machines; though, from the more general meaning assigned to the constants, they are in reality of a more general character.

Let us now apply them to the compound dynamo. Of the two forms of this machine, which, following the example of Prof. S. Thompson, we may call the Short-Shunt and the Long-Shunt respectively, we will take the former first.

Short-Shunt Compound Dynamo.



In this case $r_1 = \rho_2 = \infty$.

The conductors ρ_1 and ρ_a represent the series- and shunt-coils, and r_2 represents the external resistance.

Hence

$$R'_1 = \rho_1, \quad R'_2 = r_2,$$

$$R = r_a + \frac{\rho_a(r_2 + \rho_1)}{r_2 + \rho_1 + \rho_a} = \frac{(r_2 + \rho_1)(r_a + \rho_a) + r_a \rho_a}{r_2 + \rho_1 + \rho_a},$$

$$S = \left\{ s_a \frac{r_2 + \rho_1}{\rho_a} + s_1 \right\} \frac{\rho_a}{r_2 + \rho_1 + \rho_a}.$$

Neglecting k , we get

$$E = \frac{1}{\sigma} \left\{ nM - \frac{(r_2 + \rho_1)(r_a + \rho_a) + r_a \rho_a}{s_a(r_2 + \rho_1) + s_1 \rho_a} \right\}.$$

Again, since

$$\frac{e_2}{c_2 r_2} = \frac{E}{c_a R} \quad \text{and} \quad \frac{c_2}{r_2 + \rho_1} = \frac{\gamma_a}{\rho_a} = \frac{c_a}{(r_2 + \rho_1) \rho_a},$$

$$\begin{aligned} E &= e_2 \frac{r_2 + \rho_1 + \rho_a}{\rho_a} \times \frac{R}{r_2} \\ &= e_2 \frac{(r_2 + \rho_1)(r_a + \rho_a) + r_a \rho_a}{r_2 \rho_a}. \end{aligned}$$

Whence

$$e_2 = \frac{1}{\sigma} \left\{ \frac{nM r_2 \rho_a}{r_2(r_a + \rho_a) + \rho_1(r_a + \rho_a) + r_a \rho_a} - \frac{r_2 \rho_a}{r_2 s_a + (s_a \rho_1 + s_1 \rho_a)} \right\},$$

which may be written

$$e_2 = \frac{P_1}{A_1 + x_1} - \frac{Q_1}{B_1 + x_1},$$

if

$$P_1 = \frac{nM \rho_a}{\sigma \{ \rho_1(r_a + \rho_a) + r_a \rho_a \}}, \quad A_1 = \frac{r_a + \rho_a}{\rho_1(r_a + \rho_a) + r_a \rho_a}, \quad (3)$$

$$Q_1 = \frac{\rho_a}{\sigma(s_a \rho_1 + s_1 \rho_a)}, \quad B_1 = \frac{s_a}{s_a \rho_1 + s_1 \rho_a}, \quad x_1 = \frac{1}{r_2}.$$

In like manner, since $c_2 = e_2/r_2$,

$$c_2 = \frac{1}{\sigma} \left\{ \frac{nM \rho_a}{r_2(r_a + \rho_a) + \rho_1(r_a + \rho_a) + r_a \rho_a} - \frac{\rho_a}{r_2 s_a + (s_a \rho_1 + s_1 \rho_a)} \right\},$$

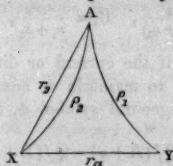
which may be written

$$c_2 = \frac{P_2}{A_2 + x_2} - \frac{Q_2}{B_2 + x_2},$$

if

$$P_2 = \frac{nM\rho_a}{\sigma(r_a + \rho_a)}, \quad A_2 = \frac{\rho_1(r_a + \rho_a) + r_a\rho_a}{r_a + \rho_a}, \quad (4)$$

$$Q_2 = \frac{\rho_a}{\sigma s_a}, \quad B_2 = \frac{s_a\rho_1 + s_1\rho_a}{s_a}, \quad x_2 = r_2.$$

Long-Shunt Compound Dynamo.

In this case

$$r_1 = \rho_a = \infty.$$

The conductors ρ_1 and ρ_2 represent the series- and shunt-coils, and r_2 represents the external resistance.

Hence

$$R'_1 = \rho_1, \quad R'_2 = r_2\rho_2 / (r_2 + \rho_2),$$

$$R = r_a + \rho_1 + \frac{r_2\rho_2}{r_2 + \rho_2} = \frac{r_2(r_a + \rho_1 + \rho_2) + \rho_2(r_a + \rho_1)}{r_2 + \rho_2},$$

$$S = s_1 + s_2 \frac{r_2}{r_2 + \rho_2} = \frac{r_2(s_1 + s_2) + s_1\rho_2}{r_2 + \rho_2};$$

whence we get

$$E = \frac{1}{\sigma} \left\{ nM - \frac{r_2(r_a + \rho_1 + \rho_2) + \rho_2(r_a + \rho_1)}{r_2(s_1 + s_2) + s_1\rho_2} \right\}.$$

Also, since

$$\frac{e_2}{\frac{r_2\rho_2}{r_2 + \rho_2}} = \frac{E}{R},$$

$$e_2 = \frac{1}{\sigma} \left\{ \frac{nMr_2\rho_2}{r_2(r_a + \rho_1 + \rho_2) + \rho_2(r_a + \rho_1)} - \frac{r_2\rho_2}{r_2(s_1 + s_2) + s_1\rho_2} \right\};$$

which may be written

$$e_2 = \frac{P_3}{A_3 + x_3} - \frac{Q_3}{B_3 + x_3},$$

if

$$P_3 = \frac{nM}{\sigma(r + \rho_1)}, \quad A_3 = \frac{r_a + \rho_1 + \rho_2}{\rho_2(r_a + \rho_1)}, \quad (5)$$

$$Q_3 = \frac{1}{\sigma s_1}, \quad B_3 = \frac{s_1 + s_2}{s_1\rho_2}, \quad x_3 = \frac{1}{r_a}.$$

And, as before,

$$c_2 = \frac{P_4}{A_4 + x_4} - \frac{Q_4}{B_4 + x_4},$$

if

$$P_4 = \frac{nM\rho_2}{\sigma(r_a + \rho_1 + \rho_2)}, \quad A_4 = \frac{\rho_2(r_a + \rho_1)}{r_a + \rho_1 + \rho_2}, \quad (6)$$

$$Q_4 = \frac{\rho_2}{\sigma(s_1 + s_2)}, \quad B_4 = \frac{s_1\rho_2}{s_1 + s_2}, \quad x_4 = r_2.$$

We thus see that if the current or difference of potential which it is required to maintain as constant as possible be represented by the general symbol ϕ , then, in the case of either machine,

$$\phi = P / (A + x) - Q / (B + x);$$

where A , B , P , and Q do not depend upon the variable resistance, and where x is either the resistance or the conductivity of the variable conductor.

All problems on self-regulation in the case of either machine may therefore be solved by means of this formula, and the application of the results is made by giving to the constants the values proper to the machine, and to the meaning of the symbol ϕ in the particular case under consideration.

It will be observed that the constant A is always either the resistance, or the reciprocal of the resistance offered by the machine to a current produced by a battery placed in the external circuit—i. e. it is either the resistance or the conductivity of the machine. If that resistance be represented by R_m , $A = R_m$ if ϕ is the external current, and $A = 1 / R_m$ if it is the external electromotive force.

If μ and m are respectively the largest and smallest values of x between which self-regulation is aimed at, then $\mu - m$ may be called the *range* of x . There will in general be some particular value of x which will be most frequently employed. This may be called the *usual value* of x , and may be denoted by ξ . As it is sometimes necessary to distinguish between the cases in which x is a resistance and a conductivity, the value of r_2 which corresponds to ξ will be indicated by r_2 .

Conditions of Efficiency.

Although a self-regulating machine is primarily intended to supply the same current or electromotive force whatever

(within certain limits) the resistance of the external circuit may be, it is nevertheless desirable that it should be as efficient as possible. It is well known that the efficiency is a maximum for a given value of the external resistance; and it is evident that the machine should not only have a high efficiency, but that the maximum should, if possible, be attained under the ordinary conditions of working.

1. Short-Shunt Machine.

Let η be the electrical efficiency. Then

$$\begin{aligned}\eta &= \frac{e_2^2}{r_2} \bigg/ \frac{E^2}{R} = \frac{\rho_a^2}{(r_2 + \rho_1 + \rho_a)^2} \times \frac{r_2}{R} \\ &= \frac{\rho_a^2}{r_2 + \rho_1 + \rho_a} \times \frac{r_2}{(r_2 + \rho_1)(r_a + \rho_a) + r_a \rho_a}.\end{aligned}$$

Taking the logarithms of both sides and differentiating with respect to r_2 , we get

$$\frac{1}{\eta} \frac{d\eta}{dr_2} = \frac{1}{r_2} - \frac{1}{r_2 + \rho_1 + \rho_a} - \frac{r_a + \rho_a}{(r_2 + \rho_1)(r_a + \rho_a) + r_a \rho_a},$$

which vanishes if

$$r_2^2(r_a + \rho_a) = (\rho_1 + \rho_a) \{ \rho_1(r_a + \rho_a) + r_a \rho_a \}. \quad (7)$$

Let r'_2 be the positive root of this equation. Hence, since

$$(r'_2 + \rho_1)(r_a + \rho_a) + r_a \rho_a = (r'_2 + \rho_1) \rho_a + r_a \{ r'_2 + \rho_1 + \rho_a \},$$

and from (7),

$$r_a \{ r'^2_2 - (\rho_1 + \rho_a)^2 \} = \rho_a \{ \rho_1(\rho_1 + \rho_a) - r'^2_2 \},$$

or

$$r_a \{ r'_2 + \rho_1 + \rho_a \} = \rho_a \frac{\rho_1(\rho_1 + \rho_a) - r'^2_2}{r'_2 - (\rho_1 + \rho_a)},$$

we have

$$\begin{aligned}& (r'_2 + \rho_1)(r_a + \rho_a) + r_a \rho_a \\ &= \rho_a \left\{ r'_2 + \rho_1 + \frac{\rho_1(\rho_1 + \rho_a) - r'^2_2}{r'_2 - (\rho_1 + \rho_a)} \right\} \\ &= - \frac{r'^2_2 \rho_a^2}{r'_2 - (\rho_1 + \rho_a)}.\end{aligned}$$

Substituting this in the expression for η , we get

$$\eta = \frac{\rho_1 + \rho_a - r'^2_2}{\rho_1 + \rho_a + r'^2_2}, \text{ or } \rho_1 + \rho_a = r'^2_2(1 + \eta)/(1 - \eta).$$

Hence, from (7),

$$\frac{r_a + \rho_a}{\rho_1(r_a + \rho_a) + r_a \rho_a} = \frac{1 + \eta}{1 - \eta} \times \frac{1}{r'_2};$$

and from (3) and (4),

$$A_1 = \frac{1 + \eta}{1 - \eta} \xi \quad \text{and} \quad A_2 = \frac{1 - \eta}{1 + \eta} \xi, \quad \dots \quad (8)$$

if we suppose r'_2 to be $= \bar{r}_2$, i. e. if the machine is so designed that the usual value of the external resistance is that which gives the maximum efficiency.

2. Long-Shunt Machine.

In this case

$$\begin{aligned} \eta &= \frac{e_2^2}{r_2} \bigg/ \frac{E^2}{R} = \frac{\rho_2^2}{(r_2 + \rho_2)^2} \times \frac{r_2}{R} \\ &= \frac{\rho_2^2}{r_2 + \rho_2} \times \frac{r_2}{r_2(r_a + \rho_1 + \rho_2) + \rho_2(r_a + \rho_1)}; \end{aligned}$$

and, as before, $d\eta/dr_2 = 0$ if

$$\frac{1}{r'_2} - \frac{1}{r'_2 + \rho_2} - \frac{r_a + \rho_1 + \rho_2}{r'_2(r_a + \rho_1 + \rho_2) + \rho_2(r_a + \rho_1)} = 0;$$

$$\therefore r'_2(r_a + \rho_1 + \rho_2) = \rho_2^2(r_a + \rho_1), \quad \dots \quad (9)$$

or

$$r_a(\rho_2^2 - r'^2_2) = r'^2_2(\rho_1 + \rho_2) - \rho_1 \rho_2^2.$$

Hence

$$\begin{aligned} r_a(r'_2 + \rho_2) + r'_2(\rho_1 + \rho_2) + \rho_1 \rho_2 \\ &= \frac{r'^2_2(\rho_1 + \rho_2) - \rho_1 \rho_2^2}{\rho_2 - r'_2} + r'_2(\rho_1 + \rho_2) + \rho_1 \rho_2 \\ &= \frac{r'^2_2 \rho_2^2}{\rho_2 - r'_2}. \end{aligned}$$

Substituting in the expression for η , we obtain

$$\eta = \frac{\rho_2 - r'_2}{\rho_2 + r'_2}, \quad \text{or} \quad \rho_2 = r'_2(1 + \eta)/(1 - \eta).$$

Hence, from (5), (6), and (9), if, as before, $r'_2 = \bar{r}_2$,

$$\left. \begin{aligned} A_3 &= \frac{1 + \eta}{1 - \eta} \times \frac{1}{\bar{r}_2} = \frac{1 + \eta}{1 - \eta} \xi, \\ A_4 &= \frac{1 - \eta}{1 + \eta} \times \bar{r}_2 = \frac{1 - \eta}{1 + \eta} \xi. \end{aligned} \right\} \quad \dots \quad (10)$$

The above values for A will be required hereafter; but if we replace A by its value in terms of R_m , equations (8) and

(10) may all be summed up in the formula

$$(1-\eta)/(1+\eta) = R_m / r'_2,$$

or

$$\eta = (r'_2 - R_m) / (r'_2 + R_m).$$

In the case of the Short-Shunt Machine,

$$r'^2_2 = (\rho_1 + \rho_a) R_m \text{ from equation (7).}$$

$$\therefore \eta = (\sqrt{\rho_1 + \rho_a} - \sqrt{R_m}) / (\sqrt{\rho_1 + \rho_a} + \sqrt{R_m}).$$

In the case of the Long-Shunt Machine,

$$r'^2_2 = \rho_2 R_m, \text{ from equation (9).}$$

$$\therefore \eta = (\sqrt{\rho_2} - \sqrt{R_m}) / (\sqrt{\rho_2} + \sqrt{R_m}).$$

These formulæ may be put in another form, which is useful as it is very easy to remember. In either kind of compound machine, and in the ordinary shunt dynamo, if the resistance of the armature is infinite, it is possible for a current generated in the external circuit to pass from one extremity of the external circuit to the other, *i. e.* from one terminal to the other, exclusively through wires which form parts of the magnetizing spirals. In the case of the Short-Shunt Machine this path includes both spirals; in the ordinary Shunt or in the Long-Shunt Compound Machine it includes one spiral only. If we call the resistance of this path Σ , then in all three cases the maximum efficiency is given by the equation

$$\eta = (\sqrt{\Sigma} - \sqrt{R_m}) / (\sqrt{\Sigma} + \sqrt{R_m});$$

and the value of the external resistance for which the maximum efficiency is attained by

$$r'^2_2 = \Sigma R_m.$$

In cases where the resistance of the shunt is large, approximate expressions may be deduced from the above by substituting the resistance of the shunt for Σ , and the sum of the resistances of the armature and series-coil for R_m .

If the maximum efficiency is attained when r_2 has its usual value, we may put the last expression in a form which will be useful hereafter. Let Y be a quantity which has with regard to Σ the same meaning as x with regard to r_2 ; *i. e.* let it be $= \Sigma$ or $1/\Sigma$ according as ϕ represents the external current or external electromotive force. Then

$$\eta = (\sqrt{Y} - \sqrt{A}) / (\sqrt{Y} + \sqrt{A});$$

and if the maximum efficiency is attained for the usual value of x ,

$$\xi^2 = YA.$$

Conditions of Maximum Power.

Since the power of a dynamo is expressed by either of the formulæ e_2^2/r_2 or $c_2^2 r_2$, it is in all cases given by $\phi^2 x$.

This has a critical value if

$$2x \frac{d\phi}{dx} + \phi = 0;$$

i. e. if

$$2x \left\{ -\frac{P}{(A+x)^2} + \frac{Q}{(B+x)^2} \right\} + \frac{P}{A+x} - \frac{Q}{B+x} = 0;$$

i. e. if

$$\frac{P(A-x)}{(A+x)^2} + \frac{Q(B-x)}{(B+x)^2} = 0.$$

Conditions of Self-Regulation.

Since

$$\phi = \frac{P}{A+x} - \frac{Q}{B+x},$$

$$\therefore \frac{d\phi}{dx} = -\frac{P}{(A+x)^2} + \frac{Q}{(B+x)^2}.$$

Hence ϕ has two critical values corresponding to the values of x given by the equation

$$x(\sqrt{P} \mp \sqrt{Q}) = -B \sqrt{P} \pm A \sqrt{Q}.$$

If we take the lower signs, x is necessarily negative unless B is negative; which can only be the case if some of the inducing spirals are wound in the negative direction, i. e. so that they reduce the strength of the magnetic field. If B is negative, it is evident that ϕ is infinite for a positive value of x . This unintelligible result is explained by reference to the general equation in which both k and σ are included, viz.

$$(E - nk)(E + R/\sigma S) = nME/\sigma.$$

For if some of the quantities s_a , s_1 , or s_2 are negative, and if the value of r_2 is such that $S=0$, then $E=nk$. The approximate expression which is being discussed fails therefore in this case, but it applies to cases in which one of the spirals is wound in the negative direction but in which B remains

positive. Reference to the expressions for B shows that this will be the case in the Short-Shunt Machine if the series-coils are negative and if the number of turns is $< s_a \rho_1 / \rho_s$, and in the Long-Shunt Machine if the shunt-coils are negative and if $s_2 < s_1$. In other cases Q is negative also.

Confining our attention therefore to the case in which B is positive, let X be the critical value of x , and Φ the corresponding value of ϕ .

Then

$$X = (A\sqrt{Q} - B\sqrt{P}) / (\sqrt{P} - \sqrt{Q}), \quad \dots (11)$$

$$\Phi = (\sqrt{P} - \sqrt{Q})^2 / (A - B), \quad \dots (12)$$

Also

$$\frac{d^2\phi}{dx^2} = \frac{2P}{(A+x)^3} - \frac{2Q}{(B+x)^3};$$

whence, at the critical point,

$$\frac{d^2\phi}{dx^2} = -2(\sqrt{P} - \sqrt{Q})^4 / (A - B)^3 \sqrt{PQ}. \quad \dots (13)$$

Hence if $A > B$, Φ is positive and is a maximum; and if $A < B$, Φ is negative and is a minimum.

The value of X given by (11) is not necessarily positive, and hence we must consider a number of cases which differ from each other in the relative magnitudes of A , B , P , and Q .

In distinguishing between them, it is convenient to remember that

$$\phi = \frac{x(P-Q) + BP - AQ}{(A+x)(B+x)}.$$

It follows from this expression and from (11) that

(I.) If $P/Q < A/B < 1$,

ϕ is negative for all positive values of x .

(II.) If $A/B < P/Q < 1$,

ϕ is positive for values of $x < (BP - AQ) / (Q - P)$.

(III.) If $A/B < 1 < \sqrt{P}/\sqrt{Q}$, and therefore $< P/Q$,

ϕ is positive for all positive values of x .

In both these last cases ϕ has no maximum value; for in (II.) X is positive but $> (BP - AQ) / (Q - P)$, while in (III.) it is negative.

Next, taking cases in which $A - B$ is positive, we see that

(IV.) If $A/B > 1 > P/Q$,

ϕ is negative for all positive values of x .

(V.) If $A/B > P/Q > 1$,

ϕ is positive for values of $x > (AQ - BP)/(P - Q)$.

(VI.) If $P/Q > A/B > \sqrt{P}/\sqrt{Q} > 1$,

ϕ is positive for all positive values of x .

In cases (V.) and (VI.) ϕ has a maximum value corresponding to a positive value of x .

(VII.) If $\sqrt{P}/\sqrt{Q} > A/B > 1$,

ϕ is positive for all positive values of x , and X is negative, so that there is no maximum value of ϕ .

The physical meaning of these different conditions may be best understood by considering the inequalities

$$P/Q > \text{or} < 1 \text{ and } P/Q > \text{or} < A/B.$$

If ϕ represents the external electromotive force, the first of these becomes, in the case of the Short-Shunt and Long-Shunt Machines respectively,

$$nM(s_a\rho_1 + s_1\rho_a) > \text{or} < \rho_1r_a + \rho_1\rho_a + r_a\rho_a, \quad (14)$$

and

$$nMs_1 > \text{or} < r_a + \rho_1. \quad (15)$$

But, by equation (1), the condition that the velocity is greater or less than the critical speed is

$$nM > \text{or} < R/S;$$

and if, in the case of the Short-Shunt Machine, we put $r_s = 0$, this becomes for the values of R and S given above,

$$nM > \text{or} < \frac{\rho_1r_a + \rho_1\rho_a + r_a\rho_a}{s_a\rho_1 + s_1\rho_a},$$

which is identical with (14).

Similarly, in the case of the Long-Shunt Machine, we get for the same condition,

$$nM > \text{or} < \frac{r_a + \rho_1}{s_1},$$

which is identical with (15).

Hence, in these cases,

$$P/Q > \text{or} < 1$$

according as the speed is greater or less than the critical speed for the resistance of the circuit when the external resistance is zero.

In like manner, it can be shown that if ϕ represents the external current, the expression $P/Q >$ or < 1 according as the speed is greater or less than the critical speed when the external resistance is infinite.

Inasmuch, however, as $x=1/r_2$ when ϕ is the external electromotive force, and $x=r_2$ when ϕ is the external current, these results may be briefly expressed by saying that P/Q is $>$ or < 1 , according as the speed is greater or less than the critical speed when x is infinite.

Similarly it may be proved that

$P/Q >$ or $< A/B$ according as the speed is greater or less than the critical speed when x vanishes.

On referring to the conditions laid down in the seven cases distinguished above in the light of these explanations, we see that they may be readily interpreted. The constant P is proportional to the speed, and we may conceive the ratio P/Q to be gradually increased by augmenting the velocity of revolution.

Thus when $A-B$ is positive, Case IV. is that in which the velocity is less than the critical velocities when $x=0$ or ∞ .

In Case V. the velocity is greater than the critical velocity when $x=\infty$, and less than that when $x=0$.

In Cases VI. and VII. it is greater than either of these critical velocities.

The condition that there shall be a maximum value of ϕ is

$$A/B > \sqrt{P/Q} > 1.$$

As an example of how such general formulæ can be applied to a particular case, we deduce from this relation and from equations (3) that, in the case of the Short-Shunt Dynamo, the external electromotive force can only have a maximum value if

$$\frac{r_a + \rho_a}{\rho_1(r_a + \rho_a) + r_a \rho_a} \cdot \frac{s_a \rho_1 + s_1 \rho_a}{s_a} > \sqrt{\left(\frac{nM(s_a \rho_1 + s_1 \rho_a)}{\rho_1(r_a + \rho_a) + r_a \rho_a} \right)} > 1;$$

or if

$$\frac{\rho_1 + \frac{s_1}{s_a} \rho_a}{\rho_1 + \frac{r_a}{r_a + \rho_a} \cdot \rho_a} > \frac{nM s_a}{r_a + \rho_a} > \frac{\rho_1 + \frac{r_a}{r_a + \rho_a} \cdot \rho_a}{\rho_1 + \frac{s_1}{s_a} \rho_a};$$

whence it follows that we must have

$$\frac{s_1}{s_a} > \frac{r_a}{r_a + \rho_a}.$$

Similar deductions can be made in the other cases.

The fact that if P/Q be sufficiently large there is no maximum value of ϕ for any positive value of x , means that if the velocity is so great that the inducing magnets are saturated so that the total electromotive force is constant, any increase in the resistance of the external circuit must be accompanied by a decrease in the current which flows through it and an increase in the electromotive force at its extremities. In the former case $x=r_2$, and in the latter $x=1/r_2$, so that in both cases an increase in x produces a decrease in ϕ . This state of things is reached before complete saturation, viz. at the point when the prejudicial effect due to the weakening of the current produced by an increase in r_2 overbalances the advantage gained by the fact that a larger proportion of the whole current passes through the shunt. This point is reached when

$$\sqrt{P} / \sqrt{Q} = A/B.$$

We are now in a position to discuss the amount of change in ϕ produced by a given finite change in x , i. e. the self-regulating power of the dynamo.

The treatment of this question must be slightly varied, according as a maximum value of ϕ does or does not occur for a value of x intermediate to μ and m , the largest and smallest values of that quantity between which self-regulation is aimed at.

If it does not, we may write

$$\left. \begin{aligned} \frac{P}{A+m} - \frac{Q}{B+m} &= \phi_1, \\ \frac{P}{A+\mu} - \frac{Q}{B+\mu} &= \frac{\phi_1}{1+q}, \end{aligned} \right\} \dots \dots (16)$$

where ϕ_1 is the value of the electromotive force or current to be kept constant, and q is a quantity which will be smaller as the self-regulation is more perfect, and which will be positive or negative according as ϕ diminishes or increases as x increase.

If, on the other hand, X lies between μ and m , then the

equations corresponding to (16) become

$$\left. \begin{aligned} \frac{P}{A+m} - \frac{Q}{B+m} &= \frac{\Phi}{1+p'} \\ \frac{P}{A+\mu} - \frac{Q}{B+\mu} &= \frac{\Phi}{1+p''} \end{aligned} \right\} \dots (17)$$

where Φ is the maximum value of ϕ .

If the constants are so chosen that the values of ϕ corresponding to μ and m are equal, then

$$\frac{P}{A+m} - \frac{Q}{B+m} = \frac{P}{A+\mu} - \frac{Q}{B+\mu} = \frac{\Phi}{1+p}, \dots (18)$$

where Φ is the maximum value of ϕ .

If, now, the following quantities are given, viz. :—

- (1) the value of ϕ which is to be attained (ϕ_1 or Φ);
- (2) the extreme values of x (μ and m);
- (3) the percentage variation in the value of ϕ which can be allowed ($100q$ or $100p$);

(4) the usual value of x (ξ), together with the condition that the maximum efficiency shall be attained for that value ;

(5) the value of the required maximum efficiency ;
then we have four equations, viz. *either* (16) or (17) and the two equations which connect η and ξ with A and Y (see p. 104) to be satisfied by the five constants A , Y , B , P , and Q . Of these, A and Y are absolutely determined by the conditions ; but any one of the three P , Q , or B may be given any convenient value if the maximum value of Φ does not lie between μ and m . If it does, we have the additional relation

$$\Phi = (\sqrt{P} - \sqrt{Q})^2 / (A - B);$$

and if the values of both p' and p'' are assigned, there are in all five equations by which the five constants are determined. When these five quantities are known, we may equate them to their appropriate values in terms of the resistances &c. of the various parts of the machine ; and we thus obtain five equations between eight quantities, viz. :— n , M , σ , s_1 , s_a (or s_2), r_a , ρ_1 , ρ_a (or ρ_2) ; in the selection of which, therefore, there is considerable range for choice.

Although the problem of the Self-Regulation of the Compound Dynamo is solved as far as the algebra is concerned,

it is possible that the values of P, Q , &c. found for arbitrarily selected values of ϕ, μ, m , &c. may be negative, or such as it would be impossible to attain in practice.

The question as to what value of q it is physically possible to attain in any given case can only be answered if we have regard to the limitations to the values of A, B, P , and Q which apply to that case. If this be done, it is possible to obtain limits to q , that is to the perfection of self-regulation.

I shall add two examples of the use of equations (16) and (18) for this purpose.

Case III. $A < B, P > Q$.

As the values of ϕ diminish as x increases, the value which corresponds to μ must be less than that which correspond to m . Hence

$$\frac{P}{A+m} - \frac{Q}{B+m} = \phi_1,$$

$$\frac{P}{A+\mu} - \frac{Q}{B+\mu} = \frac{\phi_1}{1+q}.$$

Solving for P and Q , we get

$$P = \frac{\phi_1}{1+q} \cdot \frac{\mu-m-q(B+m)}{(A-B)(\mu-m)} (A+\mu)(A+m),$$

$$Q = \frac{\phi_1}{1+q} \cdot \frac{\mu-m-q(A+m)}{(A-B)(\mu-m)} (B+\mu)(B+m).$$

Now since $A-B$ is negative, we must, if P and Q are positive, have

$$\text{and, a fortiori,} \quad \begin{aligned} q &> (\mu-m)/(A+m), \\ &> (\mu-m)/(B+m). \end{aligned}$$

Further, since $P > Q$,

$$\begin{aligned} &\{q(B+m) - (\mu-m)\} (A+\mu)(A+m) > \\ &\quad \{q(A+m) - (\mu-m)\} (B+\mu)(B+m); \\ \therefore q(A+m)(B+m)(B-A) &< (\mu-m)\{B^2 - A^2 \\ &\quad + (\mu+m)(B-A)\}; \\ \therefore q(A+m)(B+m) &< (\mu-m)\{A+B+\mu+m\}. \end{aligned}$$

Cases V. and VI.— $A/B > \sqrt{P}/\sqrt{Q} > 1$.

Taking next the case in which the values of ϕ corresponding to μ and m are equal, we deduce from (18)

$$P = \frac{\Phi}{1+p} \frac{(A+\mu)(A+m)}{A-B},$$

$$Q = \frac{\Phi}{1+p} \frac{(B+\mu)(B+m)}{A-B}.$$

Substituting these values in the equation

$$\Phi = (\sqrt{P} - \sqrt{Q})^2 / (A-B),$$

we get

$$(A-B)^2(1+p) = \{\sqrt{(A+\mu)(A+m)} - \sqrt{(B+\mu)(B+m)}\}^2;$$

which reduces to

$$\{(A-B)p + \mu - m\}^2 = 4p(A+m)(B+\mu).$$

Solving for B , we get

$$Bp = Ap + \mu + m + 2A \pm 2\sqrt{(A+\mu)(A+m)(1+p)}.$$

Hence, since $A > B$, Ap must be greater than the right-hand side of the equation. This inequality can only hold if we take the negative sign, and if

$$2\sqrt{(A+\mu)(A+m)(1+p)} > (A+\mu) + (A+m),$$

i. e. if

$$4p(A+\mu)(A+m) > (\mu-m)^2;$$

whence

$$p > \frac{(\mu-m)^2}{4(A+\mu)(A+m)}.$$

Also, since B is positive,

$$Ap + (A+\mu) + (A+m) > 2\sqrt{(A+\mu)(A+m)(1+p)};$$

$$\therefore 1+p - 2\sqrt{1+p}\sqrt{(A+\mu)(A+m)}/A + (A+\mu)(A+m)/A^2 > \mu m/A^2;$$

$$\therefore \{\sqrt{1+p} - \sqrt{(A+\mu)(A+m)}/A\}^2 > \mu m/A^2.$$

We find, therefore, that in both these cases an inferior limit can be found to p or q , which depends only on μ , m , and A . If, therefore, the latter quantity is determined by considerations relative to the efficiency, we implicitly determine at the same time a limit to the perfection of the self-regulation.

The last case is probably important practically, and the expression obtained shows that the inferior limit to p varies, if A is large, nearly as the square of the range. It will be less as A is greater; and this statement holds good for all the other cases enumerated above, so that a large value of A is favourable to good self-regulation. On referring to equations (8) and (10), we see that for a given usual value of x a high maximum efficiency is favourable to a large value of A if ϕ is the external electromotive force, and to a small value of A if ϕ is the external current. Hence we conclude that it is more difficult to combine a high efficiency with an approximately constant external current than with an approximately constant external electromotive force.

Postscript, April 30, 1885.

Since the above paper was read before the Physical Society, I have seen a solution of the question with which it deals, published by Dr. Frölich in the current number of the *Elektrotechnische Zeitschrift*, April 1885. I should like, therefore, to add a few words as to the relations between our investigations in respect both of time and result.

The method and chief conclusions given in my own paper, including the formulæ for ϕ , q , and the method of finding limits to the perfection of the regulation, were given to my Senior Class in Technical Electricity in February last. In March I had several engagements which made special calls upon my time, and partly on this account, partly because I wished to add a discussion of the case when Q or B is negative, I did not publish my results at once.

Towards the latter end of March I received my copy of the *Elektrotechnische Zeitschrift* for that month, which contained the first part of Dr. Frölich's paper. In that he gave the formulæ for the electromotive force, current, &c. in the cases of the series, shunt, and both forms of compound dynamo, in terms of the external and other resistances. The formulæ were not reduced to any simple form like the equation in ϕ given by me; and though the paper showed that Dr. Frölich was engaged on the question of self-regulation, it did not afford any indication of his method of attacking the problem. The next day after seeing this paper I sent my own investi-

gation to the Secretary of the Physical Society. It was brought before the Council of the Society on March 28th; but as it was thought that the matter was of some interest, the reading of the paper was postponed till the first Meeting after the Easter vacation (April 25th), and I was asked to attend for the purpose of giving an account of it myself. In the interval I made some improvements in the statements as to the physical meanings of the inequalities which I described at the Meeting, and have incorporated these in the text. It is only since the paper was read that I have received the April number of the *Elektrotechnische Zeitschrift*, from which I learn that Dr. Frölich has "seit dem Abdrucke des ersten Theiles dieses Vortrages," and guided partly by theory, partly by experiment, arrived at a very satisfactory solution of the problem of obtaining a constant external electromotive force.

As far, therefore, as our work is common, I do not wish to question Dr. Frölich's undoubted right to claim prior publication. The above statement is, however, necessary to show that my results were obtained independently.

On comparing the two papers, it will be seen that I have throughout considered the problem with reference to the efficiency as well as to the self-regulation, and have given the equations a more general form, in which they apply, when the symbols are properly interpreted, either to the external electromotive force or to the external current. Dr. Frölich has dealt chiefly with the case of a constant electromotive force, in which the conditions of high efficiency and good self-regulation are in accord, and both can be secured together. To obtain the best result, if a constant current is required, will need a careful balancing of opposing requirements.

As regards, however, the practical determination of another condition which shall render the problem determinate when a maximum value of ϕ does not occur between μ and m , Dr. Frölich has advanced the question a step beyond where I left it, and has added very greatly to the interest of his paper.

He has shown that when the external electromotive force is in question, and without any reference to the value of the external resistance for which the maximum efficiency occurs, excellent results may be obtained by taking (in the above

notation) $A^2Q=B^2P$. I had previously shown the physical meaning of this condition; geometrically it amounts to choosing the arbitrary constant, so that $d\phi/dx=0$ when $x=0$. Since, in the case under discussion, $x=1/r_2$, Dr. Frölich's curves and mine are so related that the product of the abscissæ of corresponding ordinates is unity, and the tangent to my curve at the point where it meets the axis of ϕ becomes in Dr. Frölich's figures an asymptote.

As expressed by the curve between e_2 and r_2 (Dr. Frölich's), good self-regulation can only be obtained in this case for values of r_2 greater than that at which the curve may be considered to have become practically parallel to its asymptote. As expressed by the curve between e_2 and $1/r_2$ (my own), the corresponding condition is that μ shall be small enough for the curve to be at the corresponding point practically parallel to the tangent at the point for which $x=0$. Dr. Frölich has shown that these conditions are fulfilled in practice, and has thus added very materially to the importance of his paper.

It must, however, be noted that this solution has nothing absolute about it unless the efficiency is considered. If it is left out of account another constant (A) is indeterminate, and may be chosen so as to make q a minimum. For when $A^2Q=B^2P$,

$$q = \frac{\mu - m}{(A + m)(B + m)} \times \frac{AB(\mu + m) + \mu m(A + B)}{AB + \mu(A + B)}.$$

In the case under consideration we may put $m=0$, and thus write

$$q = \mu^2 / \{AB + \mu(A + B)\};$$

which shows that, among the various solutions which satisfy the condition $A^2Q=B^2P$, that will give the best self-regulation for which A and B are largest.

The condition that A shall be large implies (since $A=1/R_m$) that the resistance of the machine must be small, the necessity for which could of course be readily foreseen. The condition that B should be large leads to less obvious conclusions. Thus, in the case of the Long-Shunt Machine,

$$B_s = (s_1 + s_2) / s_1 \rho_2;$$

and to make this as large as possible, we should have s_2/s_1

large and ρ_2 small. The first of these conditions is always, the second is never, fulfilled in practice. The reason of this is obvious, viz. that the efficiency would be reduced by diminishing the resistance of the shunt. The maximum efficiency is given by the formula

$$\eta = (\sqrt{\rho_2} - \sqrt{R_m}) / (\sqrt{\rho_2} + \sqrt{R_m}),$$

which diminishes with ρ_2 . It is interesting therefore to note that a high shunt-resistance is not in itself conducive to good self-regulation; and that, within the bounds of Dr. Frölich's condition, there are still opportunities for choice by which improvements in the efficiency and regulation may be effected. Probably in practice the last adjustments will be best made by some system of experiment like that described by Dr. Frölich. The theory given above will, however, enable much to be done by a few preliminary calculations.

Since I read the paper, Prof. Silvanus Thompson has called my attention to the question as to whether my formulæ indicate the advantage of back winding in the case when a constant external current is desired. Dr. Frölich has constructed a machine on this principle, to which he was led by theory. To discuss this question fully would require an investigation of the problem when B is negative; but it may be remarked that in a case such as that discussed above, when a large value of B is desirable, it may be increased in the case of the Long-Shunt Machine (B_4) by taking s_2 negative and $< s_1$.

XVII. *On the Determination of the Heat-Capacity of a Thermometer.* By J. W. CLARK, Assistant Professor of Physics in University College, Liverpool*.

I. IN the determination of specific heats a correction should be made for that part of the thermometer which is immersed in the water of the calorimeter, as its specific heat is not the same as that of an equal volume of water. Very often this correction is reduced to a mere estimation on

* Read April 25, 1885.

account of the unknown weights of glass and mercury which constitute the immersed portion of the thermometer.

When two metals of known specific gravity are fused together, and the volume of the resulting alloy is the sum of the volumes of its two constituents, it is only necessary to know the specific gravity and volume of the piece of alloy to calculate the exact volumes of the metals comprising it. Similarly, by determining the specific gravity and volume of the thermometer, the volume of mercury which it contains can be at once determined from

$$V_1 = \frac{V(S - S_2)}{S_1 - S_2}; \quad \dots \quad (1)$$

where V_1 is the required volume of mercury in the thermometer,

V the volume of the thermometer,

S the specific gravity of the thermometer,

S_2 " " " " thermometer-glass,

S_1 " " " " of Hg.

The mean value of several very closely agreeing determinations of the specific gravity of different specimens of thermometer-glass is 3.199 for lead-glass and 2.512 for soda-glass. Should it not be known of which sort of glass the thermometer consists, it may be readily ascertained by slowly introducing the upper extremity of the instrument into the reducing-flame of a blowpipe:—Soda-glass yields a yellow flame, but lead-glass blackens, from the reduction of the oxide of lead which it contains. It may be assumed that thermometers made on the continent consist of soda-glass; those made in England are usually constructed of lead-glass.

The total volume of the thermometer (V) is obtained from its weight in air and in water.

The volume of mercury contained in the thermometer having been found by (1), the volume of that part of the thermometer which is immersed in the water of the calorimeter has next to be determined. This latter volume, less the contained volume of mercury, is the immersed volume of glass.

The total immersed volume (V_s) of the thermometer is best found from the weight (W) of the thermometer in air, and its weight (W_1) when dipped in water to the same depth as it

dipped into the water of the calorimeter ; then

$$V_3 = W_{\text{grm.}} - W_1^{\text{grm.}} \quad (2)$$

But, if the bulb be cylindrical and of the same diameter as the stem, this volume may be calculated from the measured length and diameter of the immersed portion ; for if of an irregular shape it may be found by plunging it into a burette graduated in cubic centimetres and partly filled with water ; but in both cases with a less satisfactory result than is given by (2).

The volume of the *glass* of the thermometer (V_2) immersed in the water of the calorimeter is then

$$V_2 = V_3 - V_1.$$

These volumes of glass and mercury are converted into their corresponding weights, using the mean specific gravities of lead- and soda-glass already given. The sum of the products of the weight of the immersed glass and its specific heat, and of the weight of the mercury and its specific heat, is the required water-value of the part of the thermometer dipping into the water in the calorimeter.

Regnault has given the specific heat of thermometer-glass as 0.2; but for greater accuracy it would be desirable that for this value the mean specific heat of lead or of soda thermometer-glass should be substituted according to circumstances*.

The following is an illustration of the application of the method:—

Thermometer No. 2. Soda-glass.

Weight of thermometer in air . . . = W 33.97 grm.

 " " water . . . 23.01 grm.

Volume of thermometer . . . = V 10.96 c. c.

Specific gravity of thermometer . . = S 3.098

Mean sp. gr. of soda-glass = S_2 2.512

Sp. gr. of Hg = S_1 13.6

Volume of mercury in thermometer = $V_1 = \frac{V(S - S_2)}{S_1 - S_2}$
= 0.5795 c.c.

* It may perhaps be serviceable to call attention to the *Physikalisch-Chemische Tabellen* of Landolt and Börnstein as a work which contains a most useful collection of data.

Weight of thermometer in air 33.97 grm.

Weight of thermometer with lower end im-
mersed in water to the same depth as it
was immersed in water of calorimeter } 29.73

Loss of weight = volume of immersed part
of thermometer } 4.24 c.c.

Subtract contained volume of Hg = V_1 . . . = 0.5797

to get immersed volume of thermometer-glass = 3.661 c.c.

Weight of this volume of thermometer-glass } 9.196 grm.
 3.661×2.512 =

Weight of contained Hg 0.5795×13.6 . . . = 7.881

Water-value of immersed glass 9.324×0.2 = 1.840

Water-value of immersed Hg 7.881×0.0335 = 0.2639

Total water-value of the immersed part of
the thermometer } 2.1039

To test the reliability of the method, I sacrificed the thermometer by cutting off the stem at the level at which it was immersed in the water of the calorimeter, and weighed the quantities of glass and mercury: there were 9.63 and 7.64 grms. respectively, corresponding to a water-value of 2.182, *which ascribes to the above method an error of 3.7 per cent. in the required correction.* This negligible error is due to the volume of the thermometer—consisting, not only of glass and mercury, but also of the unfilled bore of the tube—and to slight deviation of the specific gravity of the thermometer-glass from the mean specific gravity used in the calculation. It may be just worth pointing out that when a fragment of the thermometer-tube is obtained from the maker with the instrument, these errors may be avoided and the *true* water-value obtained. Probably a greater error than the above is introduced into ordinary specific-heat determinations by the evaporation of the water in the calorimeter.

II. A second method for the determination of the water-value of the immersed part of a thermometer may be employed,

but it requires that the diameter of the bore of the stem be known. To determine this a fragment of the same thermometer-tube may be obtained from the maker of the instrument, or a short piece of the upper part of the thermometer may be very easily drawn-off before the blowpipe-flame, the closed end of the fragment cut off, and the diameter of the bore measured with a microscope furnished with a micrometer eye-piece and stage-micrometer. Then the weight of mercury (w), corresponding to an increase of temperature from 0° to 100°C. , is given by

$$w = al \times 13.6, \quad (1)$$

where a is the area of the tube and l the length in centimetres of $100^\circ (t^\circ)$ on the scale of the thermometer.

The weight (W) of mercury in the bulb is then

$$W = \frac{w}{\alpha t^\circ} + w, \quad (2)$$

where α is the coefficient of apparent expansion of mercury in glass. For lead-glass $\alpha = 0.000155$, and for soda-glass 0.0001586 . Having thus determined the quantity of mercury, the quantity of glass may be found as in the first method. With a thermometer containing a known weight of mercury, this method gave a water-value for the immersed part of the instrument which was 3.2 per cent. in error. I place much less reliance upon this than upon the former method, as the apparent coefficient of expansion of glass varies more than its specific gravity. The first method is quick and simple, and the results so excellent, that this second method may be almost regarded as unnecessary; but should any case arise in which it is found to possess advantages over the first, it might be possible to determine the coefficient by observing the increase in length of the thermometer when heated in steam.

University College, Liverpool.

XVIII. *On certain Cases of Electrolytic Decomposition.* By
J. W. CLARK, *Assistant Professor of Physics in University
College, Liverpool*.*

THE atomic or molecular conditions which determine metallic or electrolytic conduction are of great interest, but seem as yet too obscure to allow of any definite general conclusions respecting their nature, beyond regarding a free motion of the particles resulting either from fusion or from solution as necessary for electrolytic conduction. Even this is not without noteworthy exceptions; for Faraday has described some binary compound liquids (SnCl_4 , AsCl_3 , &c.†) which neither conduct nor decompose; whilst, on the other hand, some compound solids are known which conduct metallically ($\text{Cu}_2\text{Se}^\ddagger$, $\text{Ag}_2\text{Se}^\ddagger$, SnS_2 , $\text{CuS}^\ddagger$, to which perhaps must be added PbO_2 , MnO_2 , and Ag_2O), and some bodies which are solid and yet conduct electrolytically ($\text{Cu}_2\text{S}^\S$, $\text{Ag}_2\text{S}^\S$). Further, zincic iodide|| neither conducts nor is decomposed when rendered fluid by heat; whilst others (HgI_2 ||, HgCl_2 ||, PbFl_2 ¶) have been considered as conducting without decomposition under the same circumstances. The nature of the conduction of the metallic sulphides is very imperfectly known**.

Whilst thinking over these facts about a year and a half ago, it seemed to me very probable that a careful study of these exceptions to the general laws of electrolytic decomposition might result in more definite conjectures respecting the condition of internal or molecular structure required for conduction and decomposition. To make such a study complete requires the determination and comparison of a number of physical constants (*e. g.* colour, conductivity, expansion, specific and latent heat, refractive index, specific inductive capacity, &c.) for substances which are normal and abnormal in their electrolytic behaviour. This cannot be completely

* Read May 23, 1885.

† Faraday, 'Experimental Researches,' vol. i.

‡ Hittorf, *Pogg. Ann.* Bd. lxxxiv. 1851. § Hittorf (*l. c.*). || Faraday.

¶ Faraday. Beetz (*Pogg. Ann.* Bd. xcii. 1854) has since shown that conduction takes place in a normal electrolytic way. Faraday discovered that the solid Plumbic Fluoride began to conduct below a red heat.

** See Faraday, 'Experimental Researches'; and Hittorf (*l. c.*).

done at present for want of data; but before passing to the consideration of those bodies to which my own investigation refers, I wish to briefly refer to a few previous papers relating to substances of remarkable electrolytic behaviour from this point of view.

Hittorf states that sulphide of silver fuses at a clear red heat, but at a temperature of 180° C. it is sufficiently soft to adapt its shape slowly to that of the surface upon which it rests, and at the ordinary temperature it is malleable: a cast stick of it can be slightly bent without fracture, and it can be cut with a knife or turned in a lathe. This substance has a very low resistance, and even at the ordinary temperatures is electrolytically decomposable by a feeble current. Hittorf (*l. c.*) has concluded, from a long and careful series of experiments upon it, that it conducts electrolytically only, and that its apparent metallic conduction results when a fine thread of metallic silver has formed between the terminals of the sulphide of silver bar.

Hittorf (*l. c.*) has discovered that cuprous sulphide also conducts electrolytically "und besitzt entweder gar keine oder ganz geringe metallische Leitung." Unlike sulphide of silver this body has a high resistance, but, like it, the conductivity increases with increased temperature. Cuprous sulphide melts at a white heat, and may be cast in the form of a rod which at the ordinary temperature is very brittle, although at a "higher" temperature it may be bent.

Hittorf (*l. c.*) points out the difficulty of proving *experimentally* that the conduction of Cu_2S and Ag_2S is *entirely* electrolytic, because the formation of copper or silver by the action of the current in the substance of the bars places their ends in true metallic communication with the battery-terminals.

Cuprous selenide and argentic selenide closely resemble their corresponding sulphur compounds to which reference has just been made, and are described as being "soft" and "slightly malleable" respectively (Watts's 'Dictionary of Chemistry'). Selenides ordinarily closely resemble sulphides in their physical and chemical relations, and are consequently regarded as being possessed of similar molecular constitutions. Hittorf, however, says of cuprous and of argentic selenide,

that "beide sind gute metallische Leiter." Their electrical behaviour is therefore of an exactly opposite nature to that of their corresponding sulphur compounds.

Faraday has noticed that conduction commences in heated electrolytes at very different degrees of liquefaction and softening.

Plumbic chloride * conducts very appreciably at a temperature far below that at which it fuses, and at which it is not noticeably soft. Beetz (*l. c.*) has shown that glass begins to conduct between 200°C . and 220°C ., and Dr. Lodge reminds me that this has recently been shown to take place at 100°C .

It appears from the behaviour of these bodies that—

(i.) *In some solid electrolytes there is a sufficient mobility of the molecules at the ordinary temperature to enable electrolytic conduction to take place; in others it is conferred by a rise of temperature which is insufficient to render the solid liquid or even soft. A rise of temperature which is insufficient to render a solid liquid, or so soft as to change its shape, is also sufficient for the equalization of strain as is shown in the annealing of glass. Both electrolytic conduction and the annealing of glass take place more readily at a high than at a low temperature.*

(ii.) *Substances of apparently similar constitution may exhibit opposite forms of electrical conduction (Cu_2Se , Ag_2Se , and Ag_3S , Cu_3S).*

The bodies to which my investigation refers are mercuric chloride and mercuric iodide, which Faraday believed to conduct metallically in the fused condition.

Beetz, as I have recently found, states in a paper, "Ueber die Leitungsfähigkeit für Elektrizität welche Isolatoren durch Temperaturerhöhung annehmen" (*l. c.*), that he has obtained evidence of the electrolytic decomposition of fused mercuric iodide; and he attributes its apparent conduction without decomposition to recombination of the products. So far as I am aware, no attempt has been made to examine the nature of the conduction of fused mercuric chloride since Faraday concluded that it probably conducted metallically.

* E. Wiedemann, *Ber. d. Kgl. Sachs. Gesellschaft der Wissenschaften*, 1874. I have unfortunately been unable to refer to this paper; but since the above was written the author has kindly referred me to *Die Elektrizitätslehrer* (G. Wiedemann), Bd. i. S. 558, wherein it is stated that the iodide and bromide of lead behave similarly.

Mercuric Iodide.

Mercuric iodide is dimorphic, and at the ordinary temperature forms a scarlet powder which at 110° C. becomes yellow, and at that temperature acquires a very slight electrical conductivity (Beetz). Mercuric iodide melts at 247° C. and boils at 342° C. It is an interesting substance on account of the ease with which it volatilizes at temperatures much below its melting-point; and it is not unlikely that the ease with which the molecules are thus shown to be leaving the solid substance may be connected with a high diffusive rate when it is fused, and this may partially explain the readiness with which the products of its electrolytic decomposition often mix and recombine, thus simulating conduction without decomposition. The mercuric iodide which I have used in the following experiments was prepared either by the precipitation of recrystallized and sublimed mercuric chloride with pure potassic iodide and sublimation of the product, or by the sublimation of the commercially pure substance.

It may not perhaps be out of place to point out, in the first instance, the effect of heat upon mercuric iodide; as it is sometimes stated in text-books of Chemistry that, when heated, it undergoes partial dissociation with liberation of iodine, where the edge of the liquid is in contact with the hot glass vessel. I have, however, convinced myself that this statement applies only to the commercially pure substance, which on sublimation leaves a little impure oxide of iron, and which, on the edges of the glass vessel in which fusion and sublimation are effected, may decompose, yielding free iodine vapour. Pure mercuric iodide may be fused and sublimed without undergoing any such change; nor have I any reason to believe that, when strongly heated in a sealed glass tube, iodine is ever set free, a slight darkening in the colour of the fused substance being the only apparent alteration which it then undergoes.

Here, too, perhaps it may be convenient to describe the preparation of the graphite electrodes, which for some years past I have found very convenient for the decomposition of such substances as act upon platinum. These electrodes are best made from the "leads" of Rowney's HH cedar pencils, which may be easily removed after a few hours' soaking in

water has softened the glue sufficiently to allow of the pencil being split in half. The "lead" is then removed with a knife, and only requires heating to bright redness in a Bunsen-flame (to get rid of the shellac (?) which it contains) to render it fit for use. A platinum wire twisted or bound round one end makes a good connection for the battery-terminals.

When pure mercuric iodide in a sealed glass tube is kept in a state of fusion over a gas-flame, and electrolyzed by means of two platinum-wire electrodes passing through its ends, evidence of its decomposition may be obtained from the iodine set free about the + pole, although no mercury is discoverable at the negative. The quantity of iodine thus set free is, however, small, and does not usually seem to increase with the length of time that the current is allowed to pass; whilst the decomposition, judging from the liberation of iodine vapour about the positive or upper electrode in the tube, appears to take place at temperatures very little above the solidifying-point of the liquid mercuric iodide.

Electrolysis of this substance kept fused in narrow V- and W-shaped tubes over a gas-flame yields much the same results; but in such tubes the resistance is very high, and it is moreover difficult to ascertain precisely what is taking place within them. I therefore adopted a simple V-shaped glass tube of $\frac{3}{8}$ ths of an inch diameter and bent at an angle of about 30° , into which was placed a sufficient quantity of mercuric iodide for the experiment. The substance was then kept fused over a gas-flame, and the two graphite electrodes introduced, one at each end of the tube. In such a tube the behaviour of the substance is easily observed; and by blowing air dried over chloride of calcium into the end, the issuing vapours are readily tested for iodine with starch-paper. Under these conditions, I found that when the substance was at a temperature near the melting-point iodine could usually be detected with ease, although at a higher temperature none could be shown to exist in the free state. A control experiment with fused chloride of lead, using a current of the same strength and the same distance between the electrodes, gave evidence of chlorine without difficulty.

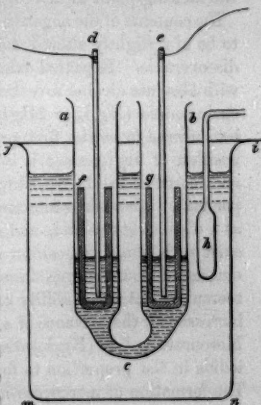
Hence it appeared possible that this substance conducted (as Ag_2S had been supposed to do) in two ways—(a) electro-

lytically, at a temperature at which it just became liquid; and (β) metallic, at higher temperatures. But during some observations upon the expansion of fused mercuric iodide in a thermometer-like tube, I noticed that the liquid underwent great contraction during cooling from a temperature a little above the melting-point, accompanied by a distinct loss of fluidity. To the latter change I at once attributed the evidence of electrolytic decomposition, which I had observed at a corresponding temperature, to be probably due, as it would be less favourable to the mixing and recombination of the products of electrolytic decomposition; and I therefore turned my attention to the construction of an apparatus in which they should be so separated as to render this less easy.

The form of apparatus which I finally adopted is shown in section in fig. 1. The tube (abc)

Fig. 1.

containing two small porous battery-pots (gf), the graphite electrode, and the mercuric iodide to be electrolyzed, is U-shaped, and the two branches are connected by a constricted portion (c), which further materially hinders the mixing of the fused products of decomposition in the two branches of the tube. This apparatus passes through a hole in the tin cover (ji) which supports it, and dips into the oil in the beaker, which is heated to the desired temperature by means of a gas-flame or sand-bath, at which it is kept constant by placing the bulb of the air-thermometer (h) in communication with a gas-regulator. When mercuric iodide

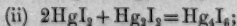
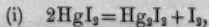


is electrolyzed in such an apparatus with a current of about 0.02 ampere, iodine is liberated at temperatures far above the melting-point of the substance. If the current was too strong the circuit was usually broken, apparently in consequence of the heat generated at the + electrode volatilizing, or other-

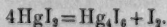
wise causing the mercuric iodide to disappear from the porous pot. After passing the current for some hours the gas was extinguished, and when the apparatus had become cool the two branches of the U-tube were cut asunder and broken open for examination. In the + branch, both within and without the porous pot, mercuric iodide and black feathery streaks of iodine were found which gave the starch reaction, and the upper part of the glass tube was coated with volatilized iodine.

Iodine dissolves freely in fused mercuric iodide, producing but little change in the colour of the latter; but just as solidification commences, more or less complete separation seems to take place with the formation of these black patches rich in iodine (and Hg_4I_6 ?) and an evolution of iodine vapour, which in the previous experiments also helped to lead to the conclusion that electrolytic decomposition occurred only near the melting-point of mercuric iodide.

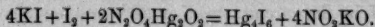
The contents of the negative branch and porous pot were found to be of a slightly altered colour; but no free mercury was discoverable. Repeated exhaustion of its pulverized contents with absolute alcohol revealed the presence of mercurous-mercuric iodide (Hg_4I_6 or $2\text{HgI}_2, \text{Hg}_2\text{I}_2$), which might perhaps be inferred to be the first product of the electrolytic decomposition of HgI_2 , since it is formed when metallic mercury and mercuric iodide are titrated together in the proper proportions. Mercurous-mercuric iodide is ordinarily regarded as a distinct compound, and I suppose rightly so; for it can exist without decomposition at a temperature at which mercurous iodide undergoes decomposition with liberation of free mercury; but the stability of this body seems to be greatly increased by the presence of a slight excess of mercuric iodide. Mercurous iodide (Hg_2I_2) dissolves readily in fused mercuric iodide in the proportion to form mercurous-mercuric iodide. The formation of mercurous-mercuric iodide by the action of the current upon mercuric iodide may therefore be represented by



or, considering it to take place in one step,

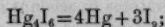


The action being slow I repeated the experiment, replacing the mercuric iodide in the apparatus just described by some mercurous-mercuric iodide precipitated nearly according to the equation

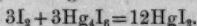


The precipitate was then washed, dried, fused, and finally sublimed from an evaporating dish on a sand-bath into a clock-glass which covered it. The substance contained a little mercuric iodide as impurity.

The products of the electrolytic decomposition of this substance are mercury and iodine, of which the former is deposited in the metallic condition, apparently according to the equation



whilst the iodine is absorbed by the Hg_4I_6 in the positive branch of the tube, forming mercuric iodide; thus,



These results may be summarized as follows:—

The conduction of fused mercuric iodide is electrolytic; but decomposition and recombination may take place so rapidly as to give rise to an apparent metallic conduction; but my investigation gives no grounds for supposing that it does not quantitatively conform to Faraday's Laws.

The causes rendering the proof of its electrolytic decomposition difficult may be summed up as follows:—

(1) *Iodine is soluble in fused mercuric iodide, and so is mercurous iodide, in the latter case with the formation of Hg_4I_6 .*

(2) *Mercuric iodide is volatile, and the presence of its vapour renders the detection of free iodine difficult by the ordinary test, and also promotes mixture in the electrolytic apparatus by distillation.*

(3) *It also seems possible that fused mercuric iodide possesses a high diffusive rate, which would further facilitate the mixture and recombination of the products of its electrolytic decomposition.*

(4) *The electrical resistance of fused HgI_2 is high.*

When pure mercuric iodide is fused over a gas-flame in a straight glass tube, of about 6 centim. in length and 0.5 centim. in breadth, and electrolyzed between platinum wire or graphite electrodes with a current of about 0.20 ampere, the resistance

shows some remarkable changes. Thus:—In an experiment which I copy from my laboratory journal, the resistance of the mercuric iodide in a sealed tube decreased as the temperature rose, until the needle of the tangent-galvanometer which was included in the circuit stood at 20° ; and on allowing the temperature to rise still higher, it fell to 9° . During cooling the inverse change occurred; for on extinguishing the gas and allowing the tube to cool, the needle advanced from 9° to 20° or 21° , and then fell gradually to 0° as conduction ceased. The cause of this change in the resistance is not very clear. I have assured myself that it is not due to any impurity in the mercuric iodide employed, nor is there a sufficient change in polarization of the electrodes at the different temperatures to account for it. With a very feeble current these resistance-changes are not marked, and indeed may escape observation in an experiment such as that which I have described. I believe this effect to be due, first to the formation, and then to the dissociation or other alteration at the higher temperature of the mercurioso-mercuric iodide produced; in support of which it may be stated that mercurous iodide undergoes rapid decomposition with separation of mercury at about the same temperature at which this change of resistance takes place. Perhaps not entirely unconnected with increase of resistance is the heat generated by the current, in consequence of the transition-resistance at the common surface of the + electrode and fused mercuric iodide, which may occasion the formation of vapour on the surface of the electrode. I have observed this give rise to a crepitating noise, and to the formation of a wave-motion spreading from the + electrode over the surface of the fused substance, and, under favourable circumstances, becoming so marked as to throw the whole tube in which the decomposition was being effected into violent oscillation.

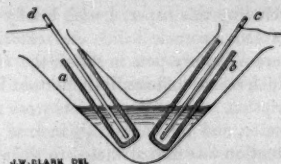
Mercuric Chloride.

Mercuric chloride far exceeds mercuric iodide in the ease with which it volatilizes at temperatures below its melting-point. It melts at 265° C. and boils at 295° C., and is more difficult to decompose with the current than mercuric iodide. This is to some extent due to its higher electrical resistance, which prevents

the use of any complex apparatus designed to represent the products of decomposition and prevent their recombination. The mercuric chloride which was used was prepared by the sublimation of the repeatedly recrystallized pure commercial salt.

The apparatus with which I have succeeded in effecting the electrolytic decomposition of fused mercuric chloride is shown in section in fig. 2, and consists of a glass tube of about $\frac{3}{8}$ ths of an inch in diameter bent at an angle of 30° . The porous pots (*ab*) and graphite electrodes (*cd*) project at each end, and the requisite temperature was supplied by the hot sand of a sand-bath. With potassic iodide and starch test-paper the evolution of chlorine at the + pole was readily detected, even when a very feeble current was employed. After the current had been passed through the fused mercuric chloride for some time, the apparatus was allowed to cool, and its contents were subsequently

Fig. 2.



pulverized and exhausted with water, which left an insoluble residue of mercurous chloride (Hg_2Cl_2), which was not entirely confined to the inside of the - porous pot.

Mercurous chloride dissolves in fused mercuric chloride, probably giving rise to a mercurioso-mercuric chloride of analogous composition to some of the well-known double chlorides which mercuric chloride forms. The want of time consequent upon the completion of an investigation on the influence of pressure on electrolytic conduction, upon which I have long been engaged, has prevented my examining this point.

The conduction of mercuric chloride is electrolytic, giving rise to chlorine and mercurous chloride; and there seems no reason to doubt that it conforms quantitatively to Faraday's Law; but the volatility of this substance, as also in the case of

mercuric iodide, and of the products of their decomposition, would render its further proof difficult.

The causes rendering its electrolytic decomposition difficult may be summed up as follows:—

(1) *The volatility of mercurous chloride and its solubility in mercuric chloride.*

(2) *The near melting- (265° C.) and boiling- (295° C.) points and great volatility of mercuric chloride facilitate the mixing and recombination of the products of its decomposition, and the vapour renders the detection of chlorine by the ordinary test difficult.*

(3) *As previously stated in reference to mercuric iodide, it is possible that the volatility of these substances is connected with a high diffusive rate, when fused, which would facilitate the recombination of the products of its electrolytic decomposition.*

(4) *The electrical resistance of the fused substance is much higher than that of the mercuric iodide.*

Before concluding this paper, I wish briefly to refer to the properties of fused mercuric iodide and chloride with reference to the porous battery-pots in which the electrodes were placed, and which seems sufficiently important to merit a few words of description. These porous pots were 2 inches long, $\frac{1}{4}$ inch in diameter, and varied from $\frac{1}{16}$ inch to $\frac{1}{8}$ thick in the walls. My attention was first attracted to their behaviour by noticing that when they were partially dipped into fused mercuric iodide, that liquid rapidly made its appearance in the pot; and the subsequent analysis of the substance surrounding the porous pots after an experiment, showed the presence of small quantities of the products of the decomposition effected by the current, such as iodine, mercurous-mercuric iodide, and mercurous chloride. Control experiments with water and with fused plumbic chloride showed that these liquids were unable to penetrate the walls of the porous pots.

The explanation of the facility with which fused mercuric iodide penetrates the walls of a porous pot and rises within it seems to be of a complex nature. The imbibition of this fused substance in the porous walls of the pot is due to capillary action, and does not account for the liquid filling the pot; for

since it is of a capillary nature, this action must cease as soon as the inner surface or wall becomes wetted. In the case of a volatile liquid such as fused mercuric iodide this action may be somewhat prolonged by its volatilization from the inner surface of the pot-wall, and by the direct formation of crystals from the vapour. The subsequent fusion of these crystals will account for the presence of some liquid mercuric iodide within the porous pot, but then this action must cease. Moreover this explanation apparently requires a difference of temperature within and without the porous pot, which, from some special experiments made upon the subject, can scarcely be assumed to exist; and I therefore think that *the explanation of the penetration of the liquid through the walls of the porous pot must be mainly sought in an easy transpiration of the fused mercuric iodide through its pores in consequence of the small initial difference of level ("head") of the liquid.*

I have already stated that analysis showed the presence of some of the products of decomposition formed within the porous pots in the undecomposed substance in which the latter were partially immersed; and this seems attributable to diffusion through the pot-walls. Little seems known respecting the rates of diffusion of fused substances through porous diaphragms; but the particular difficulties in the way of their determination for such volatile substances (which may so readily mix by distillation) seems of itself suggestive of a molecular activity not unconnected with a long free mean molecular path and of rapid diffusion.

XIX. *On the Sensitiveness of Selenium to Light, and the Development of a similar Property in Sulphur.* By SHELFORD BIDWELL, M.A., LL.B.*

THE remarkable property apparently possessed by crystalline selenium of having its electrical resistance varied by the action of light, a property which was first announced by Mr. Willoughby Smith in 1873, has been the subject of many investigations†. Of these the best known, and by far

* Read May 23 and June 13, 1885.

† Willoughby Smith, Journ. Soc. Tel. Eng. ii. p. 31; Earl of Rosse,

the most exhaustive, are the researches of Prof. W. G. Adams and Mr. R. E. Day, an account of which is published in the *Phil. Trans.* of 1877. As the result of numerous experiments, these gentlemen were led to form the opinion, that "the electrical conductivity of selenium is electrolytic"*. The principal reasons given for this conclusion are:—(1) that the resistance of the selenium-bars used appeared to depend upon the electromotive force of the battery employed, being generally diminished as the battery-power was increased; (2) that the resistance of a bar AB was generally not the same for a current in the direction AB as for a current in the direction BA; (3) that the passage of a battery-current was always followed, when the battery had been disconnected, by a secondary or polarization-current in the opposite direction, it being clearly proved that this secondary current was not due to any thermoelectric action, either in the selenium itself or in any other part of the circuit.

The authors do not, however, appear to have considered that the observed behaviour of selenium was to be explained by actual electrolysis, but rather that the molecular structure or crystalline condition of the substance was altered or modified by the action of a current of electricity in such a manner as to produce effects analogous to those which would have occurred if the selenium were an electrolyte and actually decomposed by the current. As to the possible influence of light, the following are their words †:—"Light, as we know, in the case of some bodies, tends to promote crystallization, and when it falls on the surface of such a stick of selenium, tends to promote crystallization in the exterior layers, and therefore to produce a flow of energy from within outwards, which, under certain circumstances, appears in the case of selenium to produce an electric current. The crystallization

Phil. Mag. March 1874, p. 161; *Salé*, *Proc. Roy. Soc.* 1873, p. 283; *Phil. Mag.* March 1874; *Werner Siemens*, *Phil. Mag.* November 1875, p. 416; *Draper and Moss*, '*Chemical News*,' xxxiii. p. 1; *Adams and Day*, *Proc. Roy. Soc.* 1876, p. 113; *Phil. Trans.* 1877, p. 313; *C. W. Siemens*, *Proc. Roy. Inst.* 1876, p. 68; *Sabine*, *Phil. Mag.* June 1878, p. 401; *Graham Bell*, '*Nature*,' xxii. p. 500; *Shelford Bidwell*, *Phil. Mag.* April 1881, and January 1883; *Fritts*, '*Electrical Review*,' March 7, 1885, p. 208.

* *Phil. Trans.* vol. 167, p. 328; *Proc. Roy. Soc.* 1876, p. 115.

† *Proc. Roy. Soc.* 1876, p. 117.

produced in selenium by light may also account for the diminution in the resistance of the selenium when a current from a battery is passing through it, for, in changing to the crystalline state, selenium becomes a better conductor of electricity."

Attention has lately been again directed to the subject of selenium, and its behaviour under the influence of light, by the publication, by Mr. C. E. Fritts of New York, of a new and extremely ingenious method of constructing selenium cells*. He melts a thin film of selenium upon "a plate of metal with which it will form a sort of chemical combination During the process of melting and crystallizing, the selenium is compressed between the metal plate upon which it is melted and another plate of steel or other substance with which it will not combine. . . . The non-adherent plate being removed after the cell has become cool, [he] then covers that surface with a transparent conductor of electricity, which may be a thin film of gold-leaf. . . . The whole surface of the selenium is therefore covered with a good electrical conductor, yet is practically bare to the light, which passes through the conductor to the selenium underneath." The sensitiveness to light of cells constructed in this manner seems to be far in excess of anything that has been previously obtained; and the "photoelectric" currents which (like the selenium bars of Messrs. Adams and Day) they are capable of originating, are said to be strong enough to be actually useful in practical work.

It is impossible to read Mr. Fritts's paper without being impressed by the resemblance of some of the phenomena which he describes to those of electrolysis. The mere arrangement of the apparatus—two metallic plates with a third substance between them—is in itself strongly suggestive; while the unequal resistance offered by the two surfaces, and the generation of an independent electromotive force, in conjunction with the polarization-effects above referred to†, make it

* Proc. American Assoc. 1884. Reproduced in the 'Electrical Review,' March 7, 1885, p. 208.

† "The existence of polarization," says Clerk Maxwell, "may be regarded as conclusive evidence of electrolysis." 'Electricity,' vol. i. p. 363.

hard to believe that the conduction of selenium (in the form used in experiments) is not truly and literally electrolytic.

The only considerable difficulty in the way of this hypothesis arises from the fact that selenium is not an electrolyte. Ever since its discovery in 1817, selenium has been regarded as an element, and very strong evidence indeed would be necessary to deprive it of its elementary character; this is perhaps the reason why the electrolytic theory has not previously been proposed. But there is a possible way out of the difficulty, which was suggested to me by the first words in the above quotation from Mr. Fritts's paper. He spreads the selenium upon a plate of metal *with which it will form a chemical combination*. Now selenium will, I believe, combine more or less easily with all metals, forming selenides; and in experiments upon the conductivity of selenium, it has been usual to submit the substance to prolonged heating in contact with metallic electrodes. This prolonged heating (generally followed by slow cooling) has hitherto been called "annealing;" and the undoubted fact that it diminishes the specific resistance of the selenium and increases its sensitiveness to light, has been explained by supposing that the process is favourable to perfect crystallization.

I venture to suggest, as the true explanation of the effect, that heating is favourable to a chemical combination between the selenium and the metal forming the electrodes, that a selenide is thus formed which completely surrounds the electrodes, and is perhaps diffused to some extent throughout the mass of the selenium*; and that the apparently improved conductivity of the selenium, together with the electrolytic phenomena which it exhibits, are to be accounted for by the existence of this selenide.

I have sometimes been tempted to think it possible that the apparent conductivity of selenium may in fact be *entirely* due to the impurities which it contains, and that perfectly pure selenium would be as good an insulator when in the crystalline form as it is in the vitreous condition. Vitreous selenium might contain a large percentage of conducting particles without sensible increase of its conductivity, but that this would not be the case with crystalline selenium, is rendered

* The selenium is necessarily for some time in a liquid state.

more than probable by the results of some experiments which I have described in a former communication*. If a conducting powder, such as graphite, is mixed with melted sulphur, even in small proportions, the mixture when cold is found to conduct electricity; while if a very large proportion of the same powder is incorporated with melted shellac, the shellac when cold remains sensibly as perfect a nonconductor as if it were pure. The explanation which I have given of these facts, and in support of which a number of experiments are quoted, is as follows:—The first mixture does not consist of a uniform structureless mass of sulphur, having particles of carbon imbedded in and completely surrounded by it: it is in fact an aggregation of little crystals of sulphur with carbon packed between them like mortar between bricks. The conduction thus takes place entirely through the carbon particles, which may be considered as extending in a series of chains from end to end of the mass. In the case of the shellac mixture, though the proportion of carbon may be larger than in the sulphur experiments, the resistance is still sensibly infinite, because the structureless shellac penetrates between and completely surrounds the carbon particles. Just in the same manner, selenium, when in the vitreous condition, would completely surround any particles of conducting selenides which it might contain; while, when the selenium was crystallized, the conducting particles would arrange themselves in the form of a network, capable of conveying a current of electricity.

Selenium which is free from impurities appears not to be an article of commerce. An analysis of samples collected by Professor Graham Bell from different parts of the world disclosed the presence of the metals iron, lead, and arsenic†, all of which would form conducting selenides. Nevertheless I thought it would be worth while to ascertain roughly the specific resistance of a piece of selenium which, since it has come into my possession, has never been in contact with metal. The selenium (which was supplied by Messrs. Hopkin and Williams) was melted in a mould built up of slips of glass,

* Phil. Mag. May 1882, p. 347.

† Paper read before the National Academy of Sciences, April 21, 1881.

crystallized and "annealed" in the usual way; but, contrary to the general practice, it was *not* fitted with metallic electrodes before annealing. A plate of crystalline selenium was thus formed, having a thickness of about 2 millim. and a superficial area of 1 square centim. The two opposite surfaces were rendered smooth and clean by rubbing them upon a flat board covered with fine glass-paper, and the plate was placed between two layers of thick tinfoil which were pressed into good contact with it by a weight of 500 grammes. When this arrangement was connected in circuit with 6 Leclanché cells and a reflecting galvanometer, a deflection was produced indicating a current of about $\frac{1}{30}$ micro-ampere. Assuming the electromotive force of the battery to have been 10 volts, the resistance of the plate would be 500 megohms; and therefore the resistance of a cubic centimetre of the selenium between opposite faces (*i. e.* its specific resistance) would be 2500 megohms. From the dimensions and resistance of a good selenium cell with copper electrodes, which I have in my possession, I calculated that the specific resistance of the selenium contained in it was about .9 megohm. Thus, so far as the result of a single rough experiment can be trusted, it appears that the conductivity of selenium which has been annealed in contact with copper is nearly 3000 times greater than that of selenium which has undergone similar treatment without the presence of a metal. Whether selenium, when perfectly pure, is altogether a non-conductor, would be an interesting question for an expert chemist to determine*. It is sufficient for the theory which I am at present advocating that its specific resistance should be very high.

By assuming the admixture with the selenium of metallic selenides, an explanation is afforded of the following facts:—

- (1) The diminished resistance produced by annealing.
- (2) The fact, first pointed out by Graham Bell, that the resistance of selenium appears to depend greatly upon the nature of the metals of which the electrodes are formed. For obtaining low resistance he recommends the use of brass in preference to platinum, and expresses his belief that the chemical action between the brass and selenium contributes to

* On more mature consideration I am inclined to think that it is *not*.

the low resistance of his cells, "by forming an intimate bond of union between the selenium and brass." *

(3) The fact observed by Adams and Day that there is generally a "diminution of resistance in the selenium as the battery-power is increased." The same phenomenon occurs in the mixtures of sulphur and carbon before referred to. It points to the existence of imperfect contact between conducting particles, the conduction partaking of the nature of disruptive discharge, and is consistent with the supposition that particles of conducting selenide are imbedded in the selenium †.

(4) The apparent production by a current through a piece of selenium of a "set of the molecules which facilitates the subsequent passage of a current in the opposite, but obstructs one in the same direction" ‡. This would be accounted for by the electrolytic deposition of selenium (from the selenide) upon the anode.

(5) The polarization-effects, which would also proceed from electrolysis.

(6) "A slight increase of temperature of a piece of annealed selenium is accompanied by a large increase of electrical resistance" §. This also occurs in the mixture of sulphur and carbon, and is explained by supposing that the heat-expansion of the medium draws apart the conducting particles contained in it, causing them to have fewer points of contact with each other, and thus increasing the resistance of the whole ||. A more considerable rise of temperature so greatly diminishes the specific resistance of the selenide (and perhaps of the selenium) as to more than counterbalance this effect; and thus it happens (as I have shown in a former communication ¶) that selenium cells have a "temperature of maximum resistance," which is generally a few degrees above the average temperature of the air.

* Lecture to American Assoc. 1880. Reprinted in 'Nature,' vol. xxii. p. 500.

† See "On the Electrical Resistance of Carbon-contacts," Proc. R. S. Feb. 1, 1883; and "On Microphonic Contacts," Journ. Soc. Tel. Eng. April 12, 1883.

‡ Adams and Day, Proc. R. S. 1876, p. 114.

§ Adams and Day, Phil. Trans. 1877, p. 342. See also Phil. Mag. Jan. 1883, p. 31.

|| Phil. Mag. May 1882, p. 351.

¶ Phil. Mag. April 1881.

(7) The resistance of prepared selenium is generally greatly diminished by the action of time. Prof. Adams found that the average resistance of a number of pieces of selenium was reduced to less than one fortieth in the course of a year*. During this period the selenium had been in contact with the metallic electrodes; and it seems possible that a larger quantity of selenide than was produced in the first instance by the process of annealing was slowly formed. This would especially occur at the "marked end," or anode, where there would naturally be a quantity of free selenium.

In the above argument it has been assumed that selenium will combine directly with any metal with which it is brought into contact, the combination being facilitated by the application of heat. In the case of such metals as copper, brass, and silver this is undoubtedly the fact. Indeed, an attempt to make a selenium cell with silver wires was attended with failure in consequence of the complete destruction of the metal after contact with the melted selenium for only two or three minutes. It is, however, questionable whether platinum (which was the metal used by Adams and Day) is, in any sensible degree, attacked by selenium either at the ordinary temperature or at that reached in the process of annealing. With sufficient heat the two substances will undoubtedly unite; and I have found that the surface of platinum-foil upon which melted selenium has been kept for an hour or two at a temperature probably of about 250° C. acquires a bluish-grey colour which may be due to selenide. But whether any appreciable quantity of selenide is formed in the ordinary preparation of crystalline selenium is a question only to be settled by the aid of refined chemical operations which I am incompetent to undertake, and in the meantime the suggested theory is left without direct confirmation.

But certain indirect evidence in support of my views has been forthcoming. Selenium is an element which, in its properties, closely resembles sulphur, and attempts have from time to time been made, hitherto without success, to develop in sulphur that peculiar sensitiveness to light which is such a remarkable characteristic of selenium. It occurred to me that if this property of selenium were really due to the acci-

* Phil. Trans. *loc. cit.* p. 348.

dental existence of metallic selenides, then the admixture with sulphur of metallic sulphides might be expected to lead to similar effects. It is not possible to "anneal" a stick of sulphur or a sulphur "cell" previously furnished with metallic electrodes, because sulphur does not, like selenium, solidify and crystallize at a higher temperature than that of its first melting-point. But if it is true that the virtue of annealing really lies in the fact that a chemical union of the two elements is promoted by the action of heat, it is clearly immaterial whether the substances are heated together before or after the formation of the cell. Sulphur containing sufficient metallic sulphide to render it a conductor of electricity might be used in the construction of a cell which might be expected to be sensitive to light without any preliminary annealing. This turned out to be actually the case.

Silver was the metal chosen for the experiments on account of the facility with which it combines with sulphur.

Cell No. 1.—Five parts of sublimed sulphur and one part of precipitated silver were heated together in a porcelain crucible for about two hours. The mixture was from time to time stirred with a glass rod and was finally allowed to settle, so that the bulk of the sulphide and any free silver which might remain fell to the bottom of the crucible. When the temperature was slightly above the melting-point the liquid sulphur, which was perfectly mobile, though black with minute suspended particles of sulphide, was poured off for use. Two wires of fine silver* were then coiled side by side around a strip of mica 50 millim. long and 27 millim. wide; the wires were about 1 millim. apart, and care was taken that they did not touch each other at any point. Some of the melted sulphur was spread evenly over one surface of the mica, the two wires being thus connected with each other through half their entire length by a thin layer of the prepared sulphur. When cold, this cell was connected in circuit with a battery and a galvanometer. It was found to conduct electricity, but its resistance was very high, being probably between 20 and 30 megohms. With the object of partially bridging over the intervals between the wires, the sulphur was melted by laying the cell upon a hot plate, and a piece of very thin

* Supplied by Messrs. Johnson and Matthey.

silver-foil, measuring 25 millim. by 10 millim., was laid upon its surface: this was probably entirely converted into sulphide before the cell was again cold. The cell was now found by a bridge-measurement to have a resistance of 900,000 ohms*. Once more it was connected with a Leclanché cell and a suitably shunted galvanometer; the deflection was noted, and a piece of magnesium wire was burnt at a short distance from the sulphur. The deflection was immediately more than doubled; and when the magnesium was extinguished, the spot of light at once returned to very nearly its original position. The effect was almost as great when a glass trough containing a saturated solution of alum was interposed between the sulphur and the burning magnesium†.

Now it is well known that the resistance of sulphide of silver is greatly diminished by heat‡, and it was therefore important to ascertain whether the effect just described was due to light or to heat. To speak more accurately—Is it an effect of *radiation* or of *temperature*? Exposure to radiation, whether visible or invisible, is of course always accompanied by a certain rise of temperature, and confusion has sometimes arisen, especially in discussing the properties of selenium, from failure to distinguish between the direct effects of radiation, and the indirect effects which are primarily due merely to a rise of temperature§. In the photographic processes it is radiation *per se* that produces the observed results: in the best known processes, the effective rays happen to be those which correspond to the most-refrangible part of the visible spectrum together with the invisible rays beyond it. But by more recently discovered methods the “obscure heat-rays,” as they are sometimes called, have been made available for photographic purposes||; and these do not act by virtue of any rise of temperature which they may cause, but exert

* This resistance was afterwards found to be very variable, and it was never the same with a direct and a reverse current.

† This cell was exhibited in action at the Meeting of the Physical Society on May 23rd, and at the Soirée of the Royal Society on June 10.

‡ Faraday, Exp. Res. §§ 432–439.

§ See Moser, Proc. Phys. Soc. 1881, p. 348.

|| Captain Abney is said to have obtained a photograph of a kettle of boiling water by means of the invisible radiations which it emitted.

direct chemical action upon the sensitized plate. Again, if a thermo-pile is exposed to radiation, an electromotive force is generated. Here, however, the effect of radiation is indirect; it acts only through the medium of the heat which it produces; and if an equal and similarly distributed amount of heat were communicated to the thermo-pile by any other means (as by conduction), exactly the same effect would follow. In an ordinary selenium cell radiation acts both directly and indirectly, tending to produce opposite effects. The direct effect of the radiation, whether it be visible or infra-red or ultra-violet, is a diminution of the resistance of the cell; at the same time the radiation slightly raises the temperature of the cell, and so indirectly tends to increase its resistance. If a selenium cell in circuit with a battery and a galvanometer is suddenly exposed, by withdrawing a screen, to the radiation of a black-hot poker, a momentary swing of the galvanometer-magnet will at first indicate a fall in the resistance; but this will be almost immediately followed by a rise which will increase up to a certain limit as the temperature of the cell becomes higher. The same kind of thing occurs when the cell is exposed to the infra-red or red portions of the spectrum; but in the latter case the temperature-effect merely diminishes, instead of overpowering, that directly due to radiation. If the bridge method is used for measuring the resistance, it may easily happen that the effect of gradually rising temperature escapes notice, a balance not being obtained until the temperature has become constant; and thus, probably, is to be explained the fact that different observers have attributed the most powerful action upon selenium to different parts of the spectrum, ranging from infra-red to greenish yellow.

The resistance of the sulphur cell which has been described, unlike that of most selenium cells, was diminished by a rise of temperature*. When in circuit with a Leclanché cell and a galvanometer, the effect of holding a nearly red-hot brass rod at a distance of 3 centim. from its surface, was a gradual fall of resistance, which in 15 seconds was indicated by 23 scale-divisions. When the rod was removed,

* This was not so with all the cells subsequently made. See description of cell no. 3 below.

the spot of light slowly returned to its original position, occupying several seconds in doing so. It is certain that the temperature of the sulphur must in this experiment have been much higher than when it was exposed to burning magnesium, with a solution of alum interposed, yet the effect was very much smaller; moreover, it was gradual instead of instantaneous.

Another experiment seems to prove conclusively that the resistance of the cell is diminished by the direct action of radiation, quite apart from any effect which may be produced by an incidental rise of temperature. On a cloudy day the cell, with the alum trough before it, was placed at a distance of 16 feet from a small window, all the other windows in the room being darkened. With the same battery and galvanometer as before, it was found that closing the window-shutter caused an instantaneous swing of the spot of light through 90 scale-divisions in the direction indicating increased resistance; and when the shutter was again opened, there was immediately an equal swing in the opposite direction. A delicate thermopile of 54 pairs, connected with an astatic reflecting-galvanometer of low resistance, was then put in the place of the sulphur cell, and the alum trough placed before the open end of the conical reflector attached to it. On opening the window-shutter, a deflection occurred indicating a current which was found by trial to be equal to that produced by the radiation of the human body at a distance of 10 ft. 6 in. It is needless to say that such a minute change of temperature as this implies was without sensible effect upon the resistance of the sulphur cell. There can then be no doubt whatever that the whole of the observed effect of the light upon the sulphur was due to the action of radiation as such, any change of resistance resulting from the incidental rise of temperature being quite inappreciable.

Cell No. 2.—This was constructed in a somewhat different manner. A piece of silver-foil was laid upon the surface of the mica before the two wires were wound round it, and instead of having prepared sulphur spread upon one face, the whole was immersed in *pure* melted sulphur for a few minutes, and then carefully drained. Before this treatment the silver wires were of course short-circuited by the foil, but the liquid

sulphur penetrated between them, forming a film of sulphide; and when cold, the resistance of the cell was about 100,000 ohms. Though this cell turned out to be somewhat less sensitive than the other, it seemed likely that, on account of its comparatively low resistance, it might be successfully used for a photophonic experiment. It was therefore connected in circuit with a battery of ten Leclanché cells and a telephone, and exposed to a rapidly interrupted beam of light. The telephone at once gave out a musical note, which was nearly as loud as that produced by a good selenium cell under similar circumstances.

The behaviour of this cell under changes of temperature was the same as that of the other.

Cell No. 3.—A mixture, consisting of equal parts of sublimed sulphur and precipitated sulphide of silver, was melted and spread on one surface of a slip of mica, around which two silver wires had been wound as before. No foil was used in this case. The resistance of this cell was diminished by radiation, but increased in a very marked manner by rise of temperature. A paraffin lamp, at a distance of 18 inches, produced a steady diminution of the resistance. When the lamp was placed at a distance of 10 inches, the galvanometer-needle first moved in a direction indicating a further fall of resistance; but after a few seconds, when the temperature began to rise, it turned in the opposite direction. On moving the lamp 6 inches nearer, there was at once a large deflection in the direction of increased resistance, the temperature-effect completely predominating over that of radiation.

Cell No. 4.—A strip of silver-leaf was attached to a glass plate by means of gold size, and the middle part of it was exposed to the vapour of boiling sulphur until both surfaces were completely blackened. The resistance of this cell was high, but for a few days it was extraordinarily sensitive, a reflected beam of sunlight instantly effecting a diminution of 80 per cent. in its resistance. About a fortnight after it was made, its sensitiveness had greatly fallen off.

All these cells resemble selenium in giving polarization-currents after being detached from the battery.

Supposing it to be true, then, that it is not in the selenium

or sulphur itself, but in certain metallic selenides or sulphides, that the sensitiveness to light is resident, does it become easier to explain the phenomenon, or, rather, to deprive it of the unique position which it has hitherto appeared to hold, and assign to it a place among a class of analogous effects?

I believe that, at all events in the case of the sulphur-silver cell, it is principally at the surface of the *electrodes* that the effects of radiation are to be looked for.

If a current of electricity is passed through a mass of sulphide of silver having silver electrodes, silver will be deposited upon the cathode and sulphur upon the anode. The accumulation of silver upon the cathode will clearly produce no appreciable effect upon the conductivity of the arrangement, and need not be considered. But sulphur has an enormously high resistance, and the deposition of a mere film of free sulphur upon the anode would be sufficient to stop the current altogether. The current is not in fact stopped, because the deposited sulphur at once combines with the silver of the anode, merely adding a new layer to the electrolyte. Thus the metal of the anode gradually combines with the sulphur of the electrolyte; and the conductivity of the arrangement will depend to a great extent upon the facility with which this combination is effected, the quantity of electricity which can pass in a given time being limited by the quantity of sulphur which is capable of uniting with the electrode in the same time.

Sulphur combines with silver far more readily than with iron. If, therefore, my views are correct, we should expect a cell with an iron anode to offer a much greater resistance than one which had an anode of silver, the material of the cathode, so long as it was a good elementary conductor of electricity, being of comparatively little importance. To test this idea, a cell was made consisting of electrodes of iron and silver imbedded in a mixture of sulphide of silver and sulphur. The cell being connected with a battery and a galvanometer, the deflection was 115 divisions when the current passed from the silver to the iron through the electrolyte, and only 4 divisions when the direction of the current was from iron to silver*.

* The resistance of the galvanometer was 3483 ohms, and it was shunted with a coil of 20 ohms. The resistance of the Leclanché cell was about 5 ohms.

The resistance was therefore nearly 30 times as great with an iron anode as with a silver anode. It is clear that this was not the result of bad contact between the iron and the electrolyte (such as was supposed by Graham Bell in the analogous case of selenium to account for the high resistance of a cell with platinum electrodes as compared with one in which the electrodes were made of brass), because such an effect would be independent of the direction of the current. Rather it seems that the resistances of the two anodes afford data for measuring the relative facilities with which sulphur combines with silver and with iron.

Assuming it to be thus experimentally proved that the resistance of a sulphur-silver cell depends largely upon the readiness with which sulphur unites with the anode, it follows that any cause which would assist this union would at the same time diminish the resistance. Now it is well known that certain chemical combinations are accelerated by the action of radiation—the explosive union of chlorine with hydrogen under the influence of sunlight being a familiar example. The question then suggests itself, Does sulphur combine with silver more readily when exposed to radiation than it otherwise would? There is, I believe, direct evidence that it does.

A glass plate, covered with silver leaf, was placed, with the silvered side downwards, over a crucible of boiling sulphur. One half of the plate was covered with a piece of black cloth, and the arrangement was exposed to bright sunshine. In a short time the visible portion of the silver was darkened, owing to its partial conversion into sulphide; the cloth was then removed, and the silver beneath it was found to be scarcely discoloured. There was a distinct line of demarcation between the two halves. The experiment was repeated with the same result.

Since this effect might possibly have been due to other causes than the action of light (such as the unequal condensation of sulphur vapour upon the covered and uncovered portions of the plate), the experiment was made in another form. A piece of silver leaf attached to glass was brushed over with a solution of sulphur in bisulphide of carbon; and in order to keep the temperature low and uniform, the silvered glass plate was immersed in a basin containing cold water,

which was placed in the sunshine. A board was laid across the top of the basin so as to shade one half of the plate, the other half being exposed to the direct rays of the sun. In a quarter of an hour the exposed portion of the silver had acquired a dark brown colour, while that which had been protected was of a pale yellow tint, the outline of the shadow of the board being sharply defined. I think we have here the strongest evidence that the combination of sulphur with silver is assisted by radiation.

But it is not perhaps necessary to assume that the effective action of light is confined entirely to the surface of one of the electrodes. If, as is commonly believed, electrolytic conduction involves a series of decompositions and recompositions throughout the electrolyte, any cause which assists either the separation or recombination (or both) of the components of the electrolyte might be expected to increase its conductivity; and it seems reasonable to suppose that the same influence which would assist the union of two substances when they have a tendency to unite would also be favourable to their separation when they have a tendency to separate. It is not impossible, therefore, that radiation, acting upon the surface of a thin layer of sulphide of silver through which an electric current is passing, might, by facilitating the molecular rearrangement of the atoms of sulphur and silver, exert a material influence upon the conductivity of the sulphide*.

So far as regards the explanation of the effect of light upon the resistance of selenium, I am aware that this paper contains little more than speculative suggestions, which are at present almost entirely unsupported by experimental evidence†. It is, however, noteworthy that these speculations led to the construction of a cell which, without containing a particle of selenium, behaved almost exactly as if it were composed of that substance. How far this may be considered to prove

* There are some experimental reasons, into which I am not at present prepared to enter, for believing that the admixture with the sulphide of a certain amount of free sulphur is necessary for the development of sensitiveness to radiation.

† It is especially desirable to ascertain experimentally whether the combination of selenium with the metals used as electrodes in selenium cells is assisted by light.

anything with regard to selenium I do not know ; but in any case the discovery of another substance possessing the same remarkable property seems in itself to be a matter of some interest.

XX. *On the Error involved in Professor Quincke's Method of Calculating Surface-Tensions from the Dimensions of Flat Drops and Bubbles.* By A. M. WORTHINGTON, M.A., Clifton, Bristol*.

IN one of a series of well-known papers (Pogg. *Annal.* vol. cxxxix. part 1 ; and Phil. Mag. April 1871) Prof. Quincke has recorded a large number of measures of flat drops and bubbles, from which he has deduced the value of the tensions, not only at the free surface of liquids, but also at the common surface of two liquids in contact.

The numerical results obtained exceed very appreciably the values of the surface-tensions deduced from observations with capillary tubes, and Prof. Quincke attributes the difference partly to the exposure of the surface of the meniscus in capillary tubes to atmospheric impurities, but chiefly to the fact that with capillary tubes the edge-angle is not zero, and that the quantity measured is not the surface-tension of the liquid, but the surface-tension multiplied by the cosine of the edge-angle. Thus a special significance has been attached to the high values obtained by the method of flat drops or bubbles, and these values have been widely copied and made the basis of numerical calculations.

The calculation of the results involves the integration of the equation to the liquid surface

$$T\left(\frac{1}{R} + \frac{1}{R}\right) = Dz.$$

Prof. Quincke starts by writing $\frac{1}{R} = 0$, i. e. by assuming that the drop or bubble may be treated not only as flat at the vertex, but also as having an infinite diameter. It is true that in § 5 of the paper referred to he himself comments on the

* Read June 13, 1885.

fact that these assumptions are only approximately correct; but he does not attempt to show that the error entailed by these two assumptions is insignificant. It is the object of this paper to show that the error is very considerable, amounting in most cases to as much as 10 per cent. of the whole value, and that when duly corrected the values obtained by this method do not appreciably exceed those obtained with capillary tubes.

Prof. Quincke's process consists in measuring by means of a cathetometer-microscope the dimensions of a large drop or bubble, such as is represented in section in figs. 1 and 2, that has been placed on, or below, a horizontal glass plate.

Fig. 1.



Fig. 2.



The dimensions measured are

- (1) The distance K of the vertex from the plate.
- (2) The distance k of the section of maximum radius from the plate.
- (3) The maximum diameter AB .

The first two measures afford the value $K-k$ of the distance between the vertex and the section of maximum radius, and the last affords the value L of the maximum radius.

The physical meaning of the assumptions made in the calculation may be explained in the following manner.

Imagine a central slab to be cut out of the drop between two parallel vertical planes at small unit distance apart, and that the slab is again cut in half at right angles to its length, so that we realize the portion $ABCDE'E$ of the diagram (fig. 3).

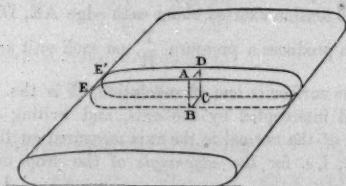
Fig. 3.



If the drop were of infinite radius, and therefore flat at the top, the slab thus obtained would be equivalent to a similar

slab cut out of a mass of liquid, shaped as in fig. 4. If we now, on this supposition, consider the equilibrium of the mass represented in fig. 3, with reference to horizontal forces parallel to its length, we can equate the hydrostatic pressure

Fig. 4.



on the rectangular end to the sum of the tensions exerted along the two edges AD and BC. Thus writing, with Quincke, $AB=K$, and writing D for the difference between the density of the drop or bubble and that of the surrounding medium,

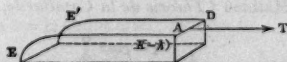
$$T + T \cos \theta = \frac{K^2 D}{2} ;$$

whence, when $\theta=0^\circ$,

$$T = \frac{K^2 D}{4}.$$

Or, if we consider the equilibrium of that portion only of the slab which lies above the horizontal section of greatest area (see fig. 5), we may equate T to the hydrostatic pressure on the rectangular area of unit breadth and depth $(K-k)$,

Fig. 5.



whence

$$T = \frac{(K-k)^2 D}{2} (1)$$

It is from this equation that Prof. Quincke calculates the value of T . It is evident that in neglecting the curvature of the vertex we are neglecting the pressure due to this curvature transmitted to the whole area $K-k$. Thus, if b be the radius of curvature in question, the pressure disregarded is

$\frac{2T}{b}(K-k)$. The surface-tension T has to balance this as well as the hydrostatic pressure due to the weight of the liquid, and neglect of this term will lead to too small a result.

Again, in neglecting the curvature in the plane at right angles to the plane of the diagram, we evidently leave out of account the tension exerted along each edge AE , DE' , of the slab, which produces a pressure $\frac{T}{R'}$ on each unit area of the surface.

Since the surface is one of revolution, R' is the length of the normal intercepted by the axis, and writing ϕ for the inclination of the normal to the axis measured on the side of the vertex, i. e. for the *edge-angle* of the drop- or bubble-forming fluid at any horizontal section, we have $\frac{1}{R'} = \frac{\sin \phi}{x}$, where x is the horizontal radius of the section; and the pressure on a horizontal strip of the rectangular end of elementary depth dz is $\frac{T \sin \phi dz}{x}$ and the total action omitted,

$$\int_0^{K-k} \frac{T \sin \phi dz}{x};$$

so that the complete equation is

$$T = \frac{(K-k)^2 D}{2} + \frac{2T}{b}(K-k) - T \int_0^{K-k} \frac{\sin \phi dz}{x} \dots (2)$$

The value of the integral of the last term to a first approximation is shown by Laplace (*Méc. Céleste*, livre x. 2^e Suppl. p. 483), or by Mathieu (*Théorie de la Capillarité*, p. 137) to be

$$\frac{4a}{3} \cdot \frac{1 - \cos^3 \frac{\phi}{2}}{x},$$

where $a = \sqrt{\frac{T}{D}}$.

When x is equal to the maximum radius L , then $\phi = 90^\circ$, and the term becomes

$$T \times \frac{4a}{3L} \left(1 - \frac{1}{2\sqrt{2}}\right).$$

We may use for (a) in this corrective term the value given by the approximate equation (1),

$$T = \frac{(K-k)^2 D}{2};$$

whence

$$\sqrt{\frac{T}{D}} = a = \frac{K-k}{\sqrt{2}}.$$

Thus the term in question reduces to

$$2T \frac{K-k}{3 \cdot 282 L};$$

and the complete equation (2) becomes

$$T = \frac{(K-k)^2}{2} D + \frac{2T}{b} (K-k) - \frac{2T(K-k)}{3 \cdot 282 L};$$

or

$$\frac{T}{D} = \frac{(K-k)^2}{2} + \frac{T}{D} 2(K-k) \left(\frac{1}{b} - \frac{1}{3 \cdot 282 L} \right).$$

Or, writing $-C$ for the value of the factor $\left(\frac{1}{b} - \frac{1}{3 \cdot 282 L} \right)$ in the corrective term,

$$\frac{T}{D} = \frac{(K-k)^2}{2(1+2K-kC)}. \quad \dots \quad (3)$$

To find the value of C we must know that of $\frac{1}{b}$. This is shown by Laplace (*loc. cit.* p. 485) or Mathieu (*loc. cit.* p. 140) to be equal to

$$2 \sqrt{2} a^{-\frac{1}{2}} \sqrt{\pi x} \tan \frac{\phi}{4} e^{-\frac{x}{a} - 4 \sin^2 \frac{\phi}{4}};$$

which, when $x=L$ and $\phi=90^\circ$, reduces to

$$\frac{1}{b} = \sqrt{\frac{8\pi L}{a^3}} \times 4142136 \times e^{-\frac{L}{a} + 585783},$$

in which we may use for a the mean value of $\frac{K-k}{\sqrt{2}}$ found by Prof. Quincke.

Before giving the numerical results, it is necessary to observe that the value of the integral $\int_{s=0}^{s=K-k} \frac{\sin \phi \, dz}{x}$ is calculated

on the assumption that the fraction $\frac{a}{L}$ is small; and, again, the value of $\frac{1}{b}$ is calculated on the assumption that $\frac{dz}{dx}$ is small, even when x is large, *i. e.* that the drop is very flat, even far from the vertex. Now the drop or bubble used by Prof. Quincke was often far from satisfying these conditions. In some cases the diameter was so small that the fraction $\frac{a}{L}$ amounted to as much as $\frac{1}{2}$, and the curvature at the vertex was so considerable that the value of $\frac{1}{b}$ cannot be satisfactorily calculated in the way described. Thus in the first measure of an air-bubble in water (Quincke, *loc. cit.* § 3, table III.) $\frac{a}{L} = .319$; while in the case of drops of petroleum in water it is as much as .3768.

I have therefore selected for correction only those measures for which the value of $\frac{a}{L}$ was smallest, though the mean value of the observed quantity $K-k$ used in calculating $\frac{1}{b}$ was generally taken from the whole of the measures given by Prof. Quincke.

Air-bubbles in Hyposulphite of Soda.

$$D = 1.1248.$$

No.	L.	K-k.	$\frac{1}{3.282 L}$	$\frac{1}{b}$	Grams per cm.	
					S. tension (Quincke).	S. tension, corrected.
3.	1.41	.3738	.2161	.0492	.07856	.06853
4.	„	.3662			.07541	
Mean (of 4 measures)					.07903	
By capillary tubes (Quincke)					.07636	

(The mean of $K-k$ used to calculate $\frac{1}{b}$ was .37485.)

Air-bubbles in Distilled Water.

D=1. Temp. 25° C.

No.	L.	K-k.	$\frac{1}{3.282 L}$	$\frac{1}{b}$	Grams per cm.		
					S. tension (Quincke).	S. tension, corrected.	
1.	.9	.4112	.3385	.3104	.08455	.08263	Rejected.
2.	1.38	.4069	.2208	.0723	.08280	.07386	
3.	1.05	.3972	.2502	.1989	.07905	.07355	
4.	1.095	.4043	.2783	.1736	.08170	.07535	
5.	1.00	.4011	.3047	.2310	.08040	.07595	
6.	1.04	.4225	.2930	.2049	.08920	.08307	
7.	1.535	.4000	.1985	.0443	.08000	.07122	
Mean (of 7)					.08253	.07550	
By capillary tubes (Quincke)						.07235 at temp. 16°-2.	
If we reject also Nos. (3), (5), and (6) on account of the largeness of the fraction $\frac{a}{L}$, the mean result is...						.07348	
The exps. of M. Wolf (see Terquem's <i>Capillarité</i> , p. 34) give						.07345 at temp. 25° C.	

Air-bubbles in Bisulphide of Carbon.

D=1.2687. Temp. 25° C.

No.	L.	K-k.	$\frac{1}{3.282 L}$	$\frac{1}{b}$	Grams per cm.	
					S. tension (Quincke).	S. tension, corrected.
1.	1.25	.2180	.2438	.0093	.03015	.027351
2.	1.35	.2230	.2226	.0046	.03157	.0287
3	1.75	.2399	.1741	0	.03651	.03369
Mean					.03274	.0299
By capillary tubes (temp. 18°) (Quincke) ...						.03343

Air-bubbles in Olive-oil.

D = .9136. Temp. 25°.8 C.

No.	L.	K-k.	$\frac{1}{3.282 L}$	$\frac{1}{b}$	Grams per cm.	
					S. tension (Quincke).	S. tension, corrected.
1.	cm. 1.5	cm. .3001	.2031	.0095	.04113	.03678
2.	1.7	.2858	.1792	.00377	.03735	.03387
3.	1.7	.2885	"	"	.03804	.03453
4.	1.56	.2905	.1953	.00719	.03850	.03470
5.	1.56	.2814	"	"	.03617	.03271
6.	1.485	.2751	.2052	.01041	.03457	.03122
7.	1.485	.2861	"	"	.03741	.03364
Mean					.03760	.03392
By capillary tubes (temp. 22°) (Quincke).						.03271

Air-bubbles in Oil of Turpentine.

D = .8867. Temp. 25°.1 C.

No.	L.	K-k.	$\frac{1}{3.282 L}$	$\frac{1}{\delta}$	Grams per cm.			
					S. tension (Quinke).	S. tension, corrected.		
3.	cm. 1.34	cm. .2665	.2274	.01199	.03149	.02818		
4.	„	.2659			.03134			
Mean (of 8 measures)					.03033	.02818		
By capillary tubes (temp. 21°·7) (Quinke) .						.02765		

(The mean value of K-k used in the calculation of $\frac{1}{b}$ is .2615 cm.)

Air-bubbles in Petroleum.

D=0.7977. Temp. 24°.2 C.

No.	L.	K-k.	$\frac{1}{3.282 L}$	$\frac{1}{b}$	Grams per cm.	
					S. tension (Quinke).	S. tension, corrected.
1.	cm. 1.475	cm. .2838	.2066	.01033	.03212	.02890
2.	1.5	.2860	.2031	.0090	.03260	.02934
3.	"	.2858				
Mean (of 3 measures)					.03244	.02912
By capillary tubes (temp. 22°-3).....						.02566

Air-bubbles in Absolute Alcohol.

D=.7906. Temp. 25°.3 C.

No.	L.	K-k.	$\frac{1}{3.282 L}$	$\frac{1}{b}$	Grams per cm.	
					S. tension (Quinke).	S. tension corrected.
1.	cm.	cm.	.1959	.00326	.02533	.02335
2.	"	.2560			.02591	
3.	1.415	.2611	.2153	.0074	.02695	.02417
4.	"	.2548			.02566	
5.	1.41	.2539	"	"	.02548	.02324
6.	"	.2560			.02590	
7.	1.44	.2573	.2116	.0063	.02616	.02383
8.	"	.2591			.02655	
					.02599	.02367
By capillary tubes at 21°·8 (Quinke)						.02237
" " " " 15° (Mendeljeff)						.02365

Mercury in Air.

D=13.5432. Temp. 20° C.

No.	L.	K-k.	$\frac{1}{3.282 L}$	$\frac{1}{\delta}$	Grams per cm.	
					S. tension (Quincke).	S. tension, corrected.
6.	1.625	.2839	.1875	.00534	.5456	.4946
7.	1.7	.2861	.1792	.00377	.5546	.5037
8.	1.65	.2822	(C calculated by propor- tional parts.)		.5624	.5102
Mean of 8 measures ...					.5503	.5028

(The mean of K-k used to calculate $\frac{1}{\delta}$ was .2863.)

Bisulphide of Carbon in Water.

D=1.2687-1.

No.	L.	K-k.	$\frac{1}{3.282 L}$	$\frac{1}{\delta}$	Grams per cm.	
					S. tension (Quincke).	S. tension, corrected.
1.	1.275	.5510	.2389	.2082	.04069	.03872
2.	"	.5478			.04021	
3.	"	.5390			.03893	
4.	1.355	.5830	.2249	.1753	.04555	.04301
5.	"	.5808			.04520	
Mean of 6 measures ...					.04256	.04086

(The mean of K-k used to calculate $\frac{1}{\delta}$ was .5583.)

Chloroform in Water.

D=0.4878.

No.	L.	K-k.	$\frac{1}{3.282 L}$	$\frac{1}{\delta}$	Grams per cm.	
					S. tension (Quincke).	S. tension, corrected.
4.	cm. 1.8	cm. .3450	.1693	.0089	.02902	.026296
5.	"	.3472			.02941	
Mean of 5 measures ...					.03010	

(The mean of K-k used to calculate $\frac{1}{\delta}$ was .35076.)

Olive-oil in Water.

$$D=1-0.9136.$$

No.	L.	K-k.	$\frac{1}{3.282 L}$	$\frac{1}{b}$	Grams per cm.	
					S. tension (Quincke).	S. tension, corrected.
6.	cm. 2.37	cm. .6941	.1286	.04185	.02082	.01818
7.	"	.6785			.01988	
Mean of 7 measures ...					.02096	

(The mean of K-k used to calculate $\frac{1}{b}$ was .69648.)

Turpentine in Water.

$$D=1-0.8867.$$

No.	L.	K-k.	$\frac{1}{3.282 L}$	$\frac{1}{b}$	Grams per cm.	
					S. tension (Quincke).	S. tension, corrected.
9.	cm. 1.15	cm. .4610	.2649	.1915	.01200	.01127
Mean of 9 measures ...					.01177	

(The mean of K-k used to calculate $\frac{1}{b}$ was .4562.)

Mercury in Aqueous Solution of Hyposulphite of Soda.

$$D=13.543-1.1248.$$

No.	L.	K-k.	$\frac{1}{3.282 L}$	$\frac{1}{b}$	Grams per cm.	
					S. tension (Quincke).	S. tension, corrected.
4.	cm. 1.365	cm. .2836	.2232	.01405	.4994	.4208
5.	"	.2730			.4628	
6.	"	.2680			.4459	
Mean of 6 measures ...					.45107	

(The mean of K-k used to calculate $\frac{1}{b}$ was .2749.)

Mercury in Water.

$$D = 13.543 - 1.$$

No.	L.	K-k.	$\frac{1}{3.282 L}$	$\frac{1}{\delta}$	Grams per cm.	
					S. tension (Quincke).	S. tension, corrected.
3.	cm. 1.5	cm. .2560	.20313	.00514	.4110	.3844
4.	"	.2605			.4256	
5.	"	.2607			.4262	
7.	"	.2527			.4005	
Mean of 7 measures ...					.4258	

(The mean of $K-k$ used to calculate $\frac{1}{\delta}$ was .26.)

Mercury in Olive-oil.

$$D = 13.543 - 0.9136.$$

No.	L.	K-k.	$\frac{1}{3.282 L}$	$\frac{1}{\delta}$	Grams per cm.	
					S. tension (Quincke).	S. tension, corrected.
3.	cm. 1.45	cm. .2362	.21014	.00306	.3523	.31066
4.	„	.2282			.3288	
Mean of 6 measures...					.3419	

(The mean of $K-k$ used to calculate $\frac{1}{\delta}$ was .2322.)

Mercury in Oil of Turpentine.

$$D = 13.543 - 0.8867.$$

No.	L.	K-k.	$\frac{1}{3.282 L}$	$\frac{1}{b}$	Grams per cm.	
					S. tension (Quinke).	S. tension, corrected.
1.	cm. 1.5	cm. .1967	.20313	.0007	.2449	.2415
2.	"	.2054			.2669	
3.	"	.2044			.2642	
4.	1.815	.1970	.20313	0	.2456	.2304
Mean ...					.2554	.236

(The mean of K-k used to calculate $\frac{1}{b}$ was .2009.)

Mercury in Petroleum.

$$D = 13.543 - 0.7977.$$

No.	L.	K-k.	$\frac{1}{3.282 L}$	$\frac{1}{b}$	Grams per cm.	
					S. tension (Quinke).	S. tension, corrected.
3.	cm. 1.9	cm. .2010	.1604	0	.2574	.23896
4.	"	.1985			.2511	
5.	1.5	.2220	.20313	.0022	.3142	.29171
6.	"	.2247			.3218	
Mean ...					.2861	.2653

(The mean of K-k used to calculate $\frac{1}{b}$ was .21155.)

Mercury in Alcohol.

$$D = 13.543 - 0.7906.$$

No.	L.	K - k.	$\frac{1}{3.282 L}$	$\frac{1}{b}$	Grams per cm.	
					S. tension (Quincke).	S. tension, corrected.
1.	cm. 1.5	cm. .2443	.20313	.0037	.3805	.3569
2.	"	.2574			.4224	
3.	"	.2422			.3740	
Mean of 6 measures ...					.4025	

(The mean of K - k used to calculate $\frac{1}{b}$ was .248.)

Olive-oil in Alcohol.

$$D = .9136 - .7906.$$

No.	L.	K - k.	$\frac{1}{3.282 L}$	$\frac{1}{b}$	Grams per cm.		
					S. tension (Quincke).	S. tension, corrected.	
3.	1.405	.1975	.21686	0	{ .00240	}	.00210
4.	..	.1872			{ .00215		
Mean of 4 measures.....							.00226

(The mean of K - k used to calculate $\frac{1}{b}$ was .1917.)

The corrected results with hyposulphite of soda, bisulphide of carbon, and petroleum do not agree satisfactorily with the measures made with capillary tubes.

The general effect of the correction is to reduce the results by about 10 per cent., but in some cases, especially of drops of one liquid in another, the value of L is so small that the accuracy of the correction may be questioned. I hope soon to be able to lay before the Society fresh determinations of the surface-tension in these cases. When L reaches or approaches 2 centims. the term $\frac{1}{b}$ is generally insignificant, though the corrective factor C, which now reduces to $\frac{1}{3.282 L}$,

is still important. In fact a drop or bubble may be considered flat before the radius can be regarded as indefinitely great.

In those cases where the corrected value still exceeds the mean value obtained by the method of capillary tubes, the difference which was previously considerable is now for the most part insignificant; witness water, olive-oil, turpentine, and alcohol. It must also be remarked that the measures of flat drops or bubbles agree among themselves far less satisfactorily than the measures of capillary elevations. Thus the numbers quoted as obtained with capillary tubes are themselves the mean of several observations, not differing from each other as a rule by more than 2 per cent., though made with tubes of various diameters; while the observations of flat drops or bubbles differ often by as much as 15 per cent., or even more.

This very variation is indeed a matter of interest, and not easily accounted for*. For though M. van der Mensbrugghe has well pointed out † that the bubble when first blown must exhibit a higher surface-tension, owing to the absorption of heat from, and consequent lowering of temperature of, the surface-layers, yet it is not easy to believe that this deviation from the normal value would be of long persistence; nor indeed do Prof. Quincke's measures always show a diminution of tension with the time.

I think, however, that we may draw the conclusion that the method of capillary tubes, when care is taken thoroughly to wet the walls above the meniscus, leads to values which are not discredited, as we had been led to think, by the results of the method of flat drops or bubbles. This is a satisfactory conclusion to come to, since the method of elevation in capillary tubes is that which has been most frequently employed for measuring the surface-tension. Prof. Quincke's value

* Mr. Bashforth (see 'An Attempt to test the Theories of Capillary Action,' p. 10) remarks on the difficulty of making any accurate direct measurement of the height $K-k$, and Prof. Guthrie has somewhere noticed the same difficulty. But Prof. Quincke's measures of $K-k$ do not show a percentage-variation much greater than that of the quantity $K\sqrt{\frac{1}{3}}$, in determining which the same difficulties do not occur.

† "Études sur les variations d'énergie potentielle des surfaces liquides," *Mém. de l'Acad. de Belgique*, t. xliii. 1878.

(.0825 gr. = 81 C.G.S. units) of the surface-tension of water has been employed by Lord Rayleigh * to calculate the wavelength in an oscillating water-jet, and leads, as he himself points out, to too low a result. Had he made use of the corrected value ($T = .0735$ gr. = 72.1 C.G.S. units), the results of p. 82, *loc. cit.*, would have been in very close agreement with theory.

In conclusion, I would remark that the great value of this particular paper of Prof. Quincke's does not, as it seems to me, depend so much on the absolute accuracy of the results presented as on the emphasis with which, at the time it was written, it represented the capillary constant to be measured as a surface-tension, and on the justice of the general considerations which are put forward, and which remain unimpeached.

A. M. WORTHINGTON.

Clifton, Bristol, June 2, 1885.

P.S. For the benefit of readers of Maxwell's 'Theory of Heat,' in which Prof. Quincke's results are given in a synoptic table, I here reproduce the table with the corrected values. Those in brackets I have been unable to check. It must be borne in mind that the numbers in the second column of tensions are still somewhat uncertain.

The values are given in grammes weight per linear metre.

Superficial Tensions at about 20° C.

Liquid.	Specific gravity.	Tension separating the liquid from		
		Air.	Water.	Mercury.
Water.....	1.0	7.35	0	38.44
Mercury.....	13.543	50.28	38.44	0
Bisulphide of Carbon	1.2687	2.99	4.09	[37.97]
Chloroform	1.4878	[3.12]	2.63	[40.71]
Alcohol	0.7906	[2.36]	...	35.69
Olive-oil.....	0.9136	3.39	1.82	31.07
Turpentine	0.8867	2.82	1.13	23.60
Petroleum	0.7977	2.91	[2.83]	26.53
Hydrochloric acid	1.1	[7.15]	...	[38.41]
Solution of Hyposulphite } of Soda	1.1248	6.85	...	42.08

* "On the Capillary Phenomena of Jets," Proc. Roy. Soc. no. 106 (1879).

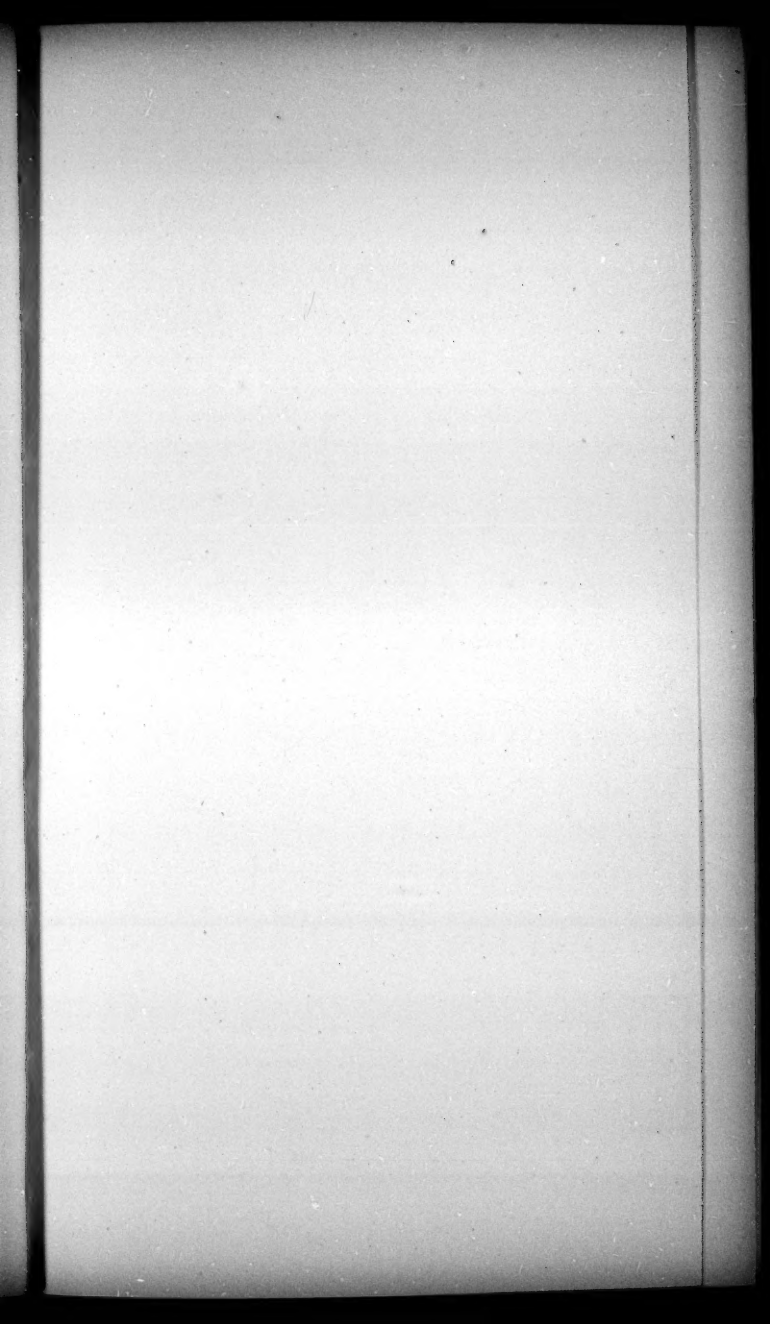


Fig. 1.

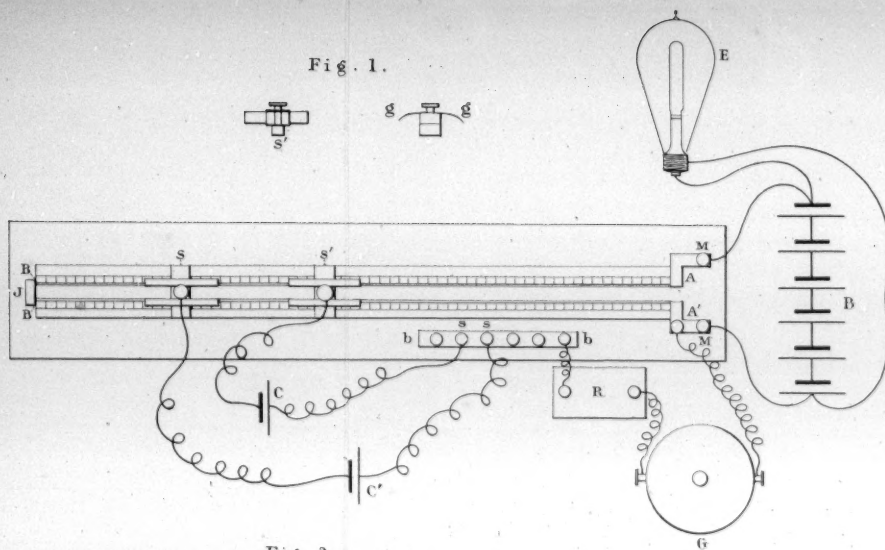


Fig. 3.

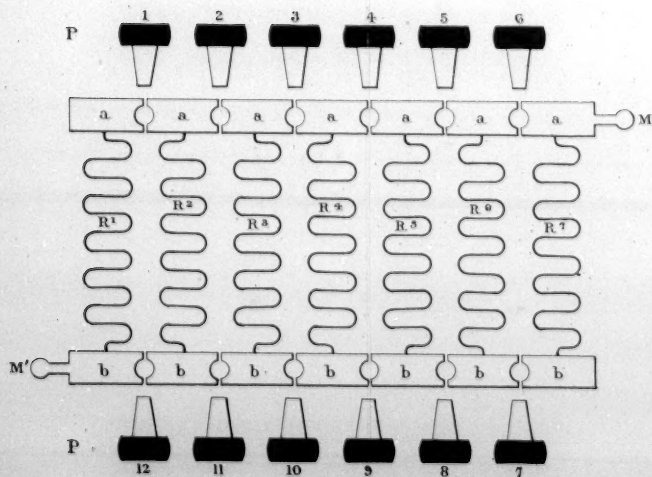


Fig. 4.



Fig. 5.

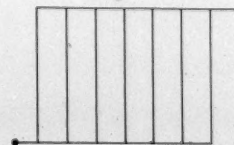
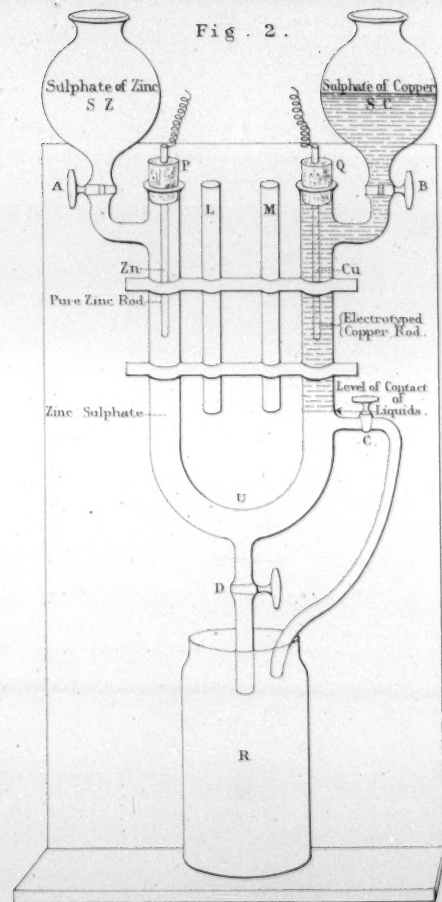


Fig. 2.



XXI. *On the Use of Daniell's Cell as a Standard of Electromotive Force.* By J. A. FLEMING, M.A., D.Sc. (Lond.), Fellow of St. John's College, Cambridge, Professor of Electrical Technology in University College, London*.

[Plate I.]

THE extensive use now being made of strong currents of electricity for purposes of illumination necessitates the employment of instruments for measuring current-strength and electromotive force of high values. The employment of electricity for telegraphic purposes, and especially in submarine telegraphy, compelled attention to a common accuracy in measurement of resistance and capacity; but for these purposes the measurements of electromotive force and currents are not of such importance as they are in the recent developments. This requirement has drawn the attention of electro-mechanicians to the subject, and led to the introduction of numerous forms of electromotive force and current-indicators, variously called potential galvanometers, voltmeters, pressure-gauges, ampere- or am-meters.

Considering first the voltmeters, they may be classified, according to Sir W. Thomson's nomenclature, into two classes—idiostatic and heterostatic. In the second class the magnetic force at a point in the neighbourhood of a conductor conveying a current is compared with another magnetic field either of the earth or of a permanent magnet, and the difference of potential at the ends of the conductor deduced therefrom. In the first class no magnetic field is made use of other than that created by the current itself.

In the case of heterostatic instruments the readings require to be controlled by a special determination of the value of the auxiliary field; and in the case of an ordinary tangent- or sine-galvanometer, the determination of the earth's horizontal magnetic force is a task requiring great care and considerable precautions. In the idiostatic instruments, in many cases, springs are used to obtain a force with which to compare the magnetic force in the neighbourhood of a conductor. The elasticity of these springs is not necessarily permanent, and is

* Read June 27, 1885.

liable to change by temperature as well as time. Accordingly in these instruments also, the constant must from time to time be determined. Pending the appearance of the new instruments promised by Sir W. Thomson, in which no springs nor permanent magnets are used, it is necessary to fall back upon galvanic cells to establish a standard of electromotive force, and the present paper is a brief collection of facts concerning the use of Daniell's cell for this purpose, and its application in the measurement of currents of various strengths. In all the experiments and methods cited below the cell has been used only to create a difference of potential, and not allowed to send any but a very small current, the means employed for comparisons being a slight modification of Clark's potentiometer, as shown in fig. 1 (Plate I.). On a stout mahogany board are fastened down two boxwood scales, each 5 feet long, divided into inches and tenths; these scales are fixed about $\frac{1}{4}$ inch apart and parallel, so as to form a groove between them. On these scales are stretched two fine uniform German-silver wires A B, A' B', about .013 inch in diameter, and having a resistance of about 1 ohm to the foot-run: one end of both wires is soldered to a thick copper junction-piece J, and the other ends respectively to copper pieces connected to terminals M, M'. The wire thus forms one length of about 10 feet stretched over two scales. This forms the potentiometer wire; its length is divided by the scale into 1200 parts, and each tenth of an inch can be divided by the eye into 10, making a possible division of 12,000. To the terminals M, M' are connected five or six large-gravity Daniell cells, and the poles of this battery are short-circuited by an Edison 16-candle lamp of 240 ohms cold resistance. This enables the battery to keep a more constant difference of potential between the terminals M, M'.

A very constant E.M.F. can be obtained by using small accumulator-cells in series: the poles being joined by a carbon-filament lamp, and the leads to the terminals M, M' taken off from the opposite sides of the lamp.

It is easy to calculate what number of cells are required to maintain a given difference of potential, say 2 volts, at the extremities of the potentiometer-wire. For let n be the

number of cells, e the E.M.F. of each, and r the internal resistance, and let R be the resistance of the lamp and potentiometer-wire combined, and v the required difference of potential desired at the terminals M, M' ; then

$$v = \frac{neR}{nr + R},$$

which determines n . If r is not known, it can be determined by a second experiment, in which v is observed in the case of a given number of cells.

The current along this German-silver wire makes a fall in potential at about the rate of 1 volt in 5 feet. On one side of the board is fastened a broad copper strap bb , having six terminals fastened upon it. Between the last of these terminals and the end of the scale-wire A' is inserted a reflecting-galvanometer of 5000 ohms resistance, and an additional resistance of 50,000 ohms R .

To the other terminals s, s on this copper strip are connected one pole (like to the pole of the battery B connected to M') of each of the cells c, c' to be compared; and the other poles of these cells are connected with sliders S, S' travelling over the wires. These sliders are blocks of wood sliding in the groove between the scales, and overhanging the wires. On them are German-silver spring-strips as shown in the figure, and which, when pressed down, make contact with the wire. The strips are backed with leather to avoid the production of thermoelectric currents. By using two or more of these sliders, it is easily seen that several cells may be balanced at once on the potentiometer; and in particular two cells c and c' can be compared in respect of electromotive force very accurately, even although the E.M.F. of the main battery is not quite constant. The introduction of the resistance R prevents any but the very smallest currents passing in the cells when the place of balance is being found on the potentiometer. The German-silver strips g, g on the sliders make contact only when pressed down; so that in the normal condition the cells c, c' are insulated. With the galvanometer in a sensitive condition it is very easy to read a difference of $\frac{1}{10000}$ of a volt on the wire, and $\frac{1}{1000}$ can be read with great accuracy.

Electromotive forces are read off directly as lengths, since the E.M.F. of the cells c, c' is directly as the distance of the constant points of their respective sliders from the end A' of the wire. Great care has to be taken in the first instance to stretch the wire uniformly, and to calibrate it if it presents any want of uniformity of resistance. Provided with this convenient means of directly comparing electromotive forces, attention was next directed to a modification of Daniell's cell suitable for the experiments required. In Dr. Alder Wright's extensive researches on the subject*, the form of cell employed was that sometimes called Raoult's form, in which two small beakers contain the two solutions, and are connected by an inverted Y-tube, the ends of which are tied over with thin bladder. This form of cell is undoubtedly convenient for some purposes, but not for others, as when the cell has to be heated or cooled for comparison at different temperatures. Moreover, the siphon becomes filled in a very short time with a mixture of solutions by diffusion; and if the levels of the liquids are not exactly equal, a siphoning action is added to that of diffusion in carrying liquid over from one side to the other. After some numerous trials of all existing forms which have been proposed for Daniell's cell, the following was devised, which, though more bulky than others, has yet given great satisfaction, and has the great recommendation of always standing ready for use.

A large U-tube, about $\frac{3}{4}$ inch diameter and 8 inches long in the limb, has four side tubes (see Pl. V. fig. 2). The two top ones, A and B, lead to two reservoirs Z and C, and the bottom ones C and D are drainage-tubes. These side tubes are closed by glass taps. The whole is mounted on a vertical board, with a pair of test-tubes between the limbs. Suppose, now, a Daniell's cell is to be formed with solutions of zinc sulphate and copper sulphate, and that the zinc-sulphate solution is the denser. The left-hand reservoir SZ is filled with the zinc solution, and the right-hand reservoir SC with the copper solution.

* "On the Determination of Chemical Affinity in Terms of Electromotive Force.—Part V. On the Relationships between the Electromotive Force of a Daniell Cell and the Chemical Affinities involved in its Action." *Proceedings of the Physical Society of London*, Vol. V. Part I. p. 44 *et seq.* [*Phil. Mag.* vol. xiii. p. 265.]

The electrodes are zinc and copper rods, Zn and Cu, passed through vulcanized-rubber corks, P and Q, fitting air-tight into the ends of the U-tube.

The operation of filling is as follows:—Open the tap A and fill the whole U-tube with the denser zinc-sulphate solution; then insert the zinc rod and fit it tightly by the rubber cork P. Now, on opening the tap C the level of the liquid will begin to fall in the right-hand limb but be retained in the closed one. As the level commences to sink in the right-hand limb, by opening the tap B copper-sulphate solution can be allowed to flow in gently to replace it; and this operation can be so conducted that the level of demarcation of the two liquids remains quite sharp, and gradually sinks to the level of the tap C. When this is the case, all taps are closed and the copper rod inserted in the right-hand limb.

Now it is impossible to stop diffusion from gradually mixing the liquids at the surface of contact; but whenever the surface of contact ceases to be sharply defined, the mixed liquid at the level of the tap C can be drawn off, and fresh solutions supplied from the reservoirs above.

In this way it is possible to maintain the solution pure and unmixed round the two electrodes with very little trouble; and the electrodes, when not in use, can be kept in the idle cells or test-tubes L and M, each in its own solution. In making experiments concerning temperature, the whole U-tube can be immersed in a vessel of water or ice up to nearly the top of the reservoirs, and the temperature in the solutions taken by means of a thermometer passing through the rubber-cork. Each of the electrodes can be removed for examination or change without in the least disturbing the surface of contact of the solutions. If experiments are being made in which the sulphate-of-copper solution is the denser, the position of the solutions is interchanged. The bottle R serves to collect the waste solutions.

The electrodes are made of rods of the purest zinc and copper, about 4 inches long and $\frac{1}{4}$ inch diameter. The zinc found most suitable is made from zinc twice distilled and cast into rods; the copper prepared by electro-depositing on a very fine copper wire, until a cylinder of the required thickness is obtained.

With these appliances to hand, all the facts recorded by previous experimenters have been carefully repeated and confirmed, and the influence of each variable upon the electromotive force examined. The results are collected as follows. For the sake of brevity, a Daniell's cell in which zinc, sulphate of zinc, sulphate of copper, and copper are the elements will be called a normal Daniell; and the statements here below refer to the variations in the difference of potential between the poles of the normal Daniell, when sending either no current or only an infinitesimal one, caused by variations in the physical state of the four elements. In each set of experiments the greatest care was taken to keep all the elements constant, except the one which was purposely being varied in order to detect the influence of it on the whole electromotive force.

1. *The Effect of Variation in the Copper Surface and Condition.*

In these experiments the copper- and zinc-sulphate solutions were sometimes of the same specific gravity, sometimes different; and in some cases the zinc electrode was a rod of the purest cast zinc amalgamated with pure mercury, sometimes a rod of commercial zinc wire cleaned but not amalgamated. Some scores of observations were made with identical zinc plate and solutions, but various kinds of copper plates*; and all observations have confirmed the conclusions of Dr. Alder Wright that the most uniform results are obtained from

* All the following values are given in real volts, taking Lord Rayleigh's value for the mean B.A. unit of resistance as equal to $\cdot 9867$ earth-quadrant per second.

The determinations have been made against a Clark cell, kindly compared by Lord Rayleigh with his No. 1 cell, whose value is given in his paper in the Phil. Trans. part ii. (1884) as $1\cdot435$ true volt at 15°C . My standard, called hereafter F_{20} , is greater than this by 3 parts in 4940, or is $1\cdot436$ volt at 15°C .; and experiments have shown that its variation-coefficient is nearly that found by Lord Rayleigh, viz. $\cdot 082$ per cent. per degree diminution of E.M.F.

Accordingly, the value of this standard Clark cell F_{20} has been taken as equal to

$$1\cdot436(1 - \cdot 00082 t^{\circ}) \text{ volt at } t^{\circ}\text{C}.,$$

and all values given in the text are in terms of this.

copper plates freshly electrotyped over with a new clean pinkish surface of virgin copper, when the plate is taken straight out of the electrotyping bath and put into the cell without any delay or touching except a slight rinse with distilled water. The result of comparing a freshly electrotyped rod with a rod scraped and glass-papered, was to show that the scraped copper rod gave a higher electromotive force by about 4 parts in 629, or 6 parts in 1000. When both rods were freshly electrotyped in the same bath, and inserted in the same cell, there was absolutely no difference between them. Cells were compared with solutions of equal density 1.2 at 15° C. and amalgamated pure cast zinc, but different copper rods, as follows :—

	E.M.F.
Electrotyped copper rod . .	1.105 volt.
Scraped copper rod . . .	1.111 „

The exact amount of difference varied slightly in other experiments ; but with equally clean surfaces, the electrotyped fresh surface has a lower electromotive force than the scraped. The effect of amalgamating the copper rod is, like electrotyping, to lower its value below that of the clean scraped, but it is not so uniform. The following are values obtained from equal copper rods in the same cell, one having an electroplated surface and the other being slightly amalgamated with pure mercury :—

Freshly electro-deposited copper rod.	Freshly amalgamated copper rod.
1.103	1.104
1.106	1.104
1.105	1.104
1.106	1.105

The mean of the values of the electro-deposited copper exceeds by about 1 in 1000 the mean of the amalgamated copper ; but if the amalgamation is done thickly it causes more serious deviations, and though more convenient is not quite so reliable as electroplating the rod with fresh copper, but is preferable to employing simply a scraped or glass-papered surface of rolled copper. A large series of determinations were next made on the effect of oxidation of the copper

surface. If a copper rod, newly electroplated, is left in the copper-sulphate solution, it gradually gets oxidized and covered with brown patches of oxide, and if amalgamated, the mercury sinks in and the surface gets brown and patchy. When this is the case the electromotive force rises, and by uncertain amounts, depending on the degree of oxidation. It is very remarkable how small a trace of brownish oxidation on the surface raises the electromotive force several parts in 1000, amounting in some cases to nearly 1 per cent. The following is one set out of many experiments on this point:—

Two solutions of zinc and copper sulphate were prepared of the pure crystals. The specific gravity of the zinc sulphate was 1.4, and that of the copper sulphate was 1.1. A rod of pure unamalgamated zinc and one of electrotyped copper was also taken. Keeping all other conditions constant, the rod of copper was first cleanly and carefully electrotyped over with a new fresh surface of copper, and exhibited no trace of oxidation; and the E.M.F. of this cell was then taken against a carefully tested standard Clark cell of known value; and the rod was then suffered to oxidize on the surface by successive exposure to the air, the E.M.F. of the cell being taken at each stage.

	E.M.F. of cell.
Copper, perfectly pure, unoxidized . .	1.072 volt.
„ slightly oxidized, brown spots.	1.076 „
„ more oxidized	1.082 „
„ covered with dark-brown oxide film	1.089 „
„ cleaned, replated with fresh pinkish electro-surface . .	1.072 „

Many other experiments of this sort showed the same thing. A Raoult cell was formed by taking two beakers, one containing zinc sulphate of specific gravity 1.2, and the other copper sulphate of the same specific gravity, and using amalgamated cast zinc and electro-deposited copper rods. The beakers were connected, when required, by a piece of clean cotton-wick dipped in water, and connecting both solutions by dipping into each. The copper pole was first freshly electroplated, and the following values obtained for the E.M.F. of the cell as it gradually oxidized:—

	E.M.F. of cell.
Freshly electro-deposited surface of copper	1.105 volt.
Slightly oxidized, faint brown tinge . . .	1.106 "
More oxidized	1.107 "
Still more oxidized	1.109 "
Re-electroplated surface	1.105 "

Hence these and many similar experiments all teach that oxidization, even the slightest trace, of the copper raises the E.M.F.; and that, in order to get the real value proper to the combination, the copper must be electrotyped over with a thin pure film of copper, and exhibit no trace of brown spots of oxide, and be used at once.

2. *The Effect of Variation in the Zinc Surface and Condition.*

Numerous experiments have been made to investigate whether there is any certain difference between zinc amalgamated or unamalgamated or cast or rolled. There is very little, if any, certain difference between perfectly pure cast zinc unamalgamated and the same amalgamated with pure mercury. There are greater differences if the zinc is not pure, and variations are introduced if impure mercury is used for amalgamation, all of them uncertain in amount.

The effect of oxidation of the zinc is to lower the electromotive force. If the bright surface of the zinc becomes tarnished, it always shows a slightly lower E.M.F. The smallest deposit of copper upon the zinc, due to diffusion of the copper salt into the zinc, is indicated by a marked depression, amounting to 2 or 3 per cent. On the whole, the only consistent values are obtained from chemically pure zinc with a bright fresh untarnished surface, whether amalgamated or not; and the best results are given with pure zinc amalgamated with pure mercury. In Dr. Alder Wright's memoir the above conclusions are enforced by tabulated results of most extensive experiments.

On the question of amalgamation of the zinc we may quote a note on the subject by M. G. Lippmann in the *Journal de Physique*. There is an opinion expressed by some authors that amalgamated zinc has a higher E.M.F. than unamalgamated, even if pure. If a plate of each of the two substances

be immersed in sulphate-of-zinc solution, a couple is formed in which amalgamated zinc forms the negative pole, inasmuch as it is the more readily acted upon by oxygen. Such, at least, is the result obtained if ordinary commercial zinc is used and ordinary sulphate-of-zinc solution. In a recent work, M. W. Robb shows that if care be taken to use electro-deposited zinc, and also zinc-sulphate solutions deprived of free acid, the element zinc, zinc sulphate, and amalgamated zinc will not manifest any appreciable E.M.F. The superiority of amalgamated zinc over ordinary zinc is easily explained; but when pure zinc is used, deprived of any oxide on the surface by slight rinsing in dilute sulphuric acid before placing it in the sulphate of zinc, there is little or no certain difference, to the extent of 1 part in 1000 in the case of the normal Daniell, between the pure zinc amalgamated or unamalgamated.

3. *The Influence of Density of the Solutions on the Electromotive Force.*

The chief reason for the differences in the values assigned by various observers to the E.M.F. of the normal Daniell is due to the great influence that the specific gravity of the two exciting solutions exerts on the resultant E.M.F. of the combination; and the general law of the effect is this—If a Daniell cell be formed of amalgamated pure zinc, freshly-electrotyped copper, and solutions of pure zinc sulphate and copper sulphate of equal specific gravity, then, taking this cell as a standard of reference, increasing the density of the zinc sulphate lowers the electromotive force, and increasing the density of the copper sulphate raises the electromotive force, within, at any rate, the limits of density from 1.01 as an inferior, up to saturation as a superior limit. Zinc sulphate saturated at 15° C. has a density of a little above 1.4, and copper sulphate similarly saturated 1.2; and Dr. Alder Wright shows by experiments that the increment and decrement of E.M.F. for identical increment or decrement of density of both solutions are so nearly equal that for equally dense solutions, within limits, the E.M.F. is independent of the absolute density of both. This fact has been confirmed by many other observers. Amongst other careful experiments may be noticed

those of Carhart*, whose experiments were specially directed towards ascertaining the influence which variation in the density of the zinc sulphate exerts on the resultant E.M.F.

In using a Daniell cell as a laboratory standard it is necessary, therefore, to prepare and standardize the solutions of zinc and copper sulphates with the same care as if they were to be used for volumetric analysis. It is a good plan to prepare two large stock bottles of solutions, and having carefully determined the density to adjust them to known specific gravity by the aid of the specific-gravity bottle or hydrometer. For general use two standard solutions of each salt are specially useful. First, a solution of sulphate of copper very nearly saturated at 15° C. and having a specific gravity of 1.200, and a solution of sulphate of zinc of equal density. Secondly, a solution of sulphate of copper of density 1.100 at 15°, and a solution of sulphate of zinc of 1.4. These last values are chosen because they were employed by Sir W. Thomson in his standard gravity-cell; and they can be used in either his form of gravity-cell, or in the U-tube form above, or in Raoult's form of separate vessels and siphon†.

A very large number of comparisons have been made of the E.M.F. of cells set up with these solutions and the E.M.F. of Clark's cells, whose value has been compared directly with cells standardized in absolute measure.

If a Daniell cell is carefully made up, either in the Raoult form or U-tube form, of solutions of pure zinc and copper sulphate not sensibly interdiffused at the level of contact, and with pure amalgamated cast zinc and freshly electrotyped copper plate, which is evenly plated with a new uniform pinkish deposit of electro-copper free from all brown spots of oxide, the E.M.F. of this cell, taken at once, is very close to 1.102 true volt, and the ratio of the E.M.F. of this cell to a cor-

* A paper read before the American Association for the Advancement of Science, Sept. 5, 1884. See 'Telegraphic Journal and Electrical Review,' vol. xv. p. 250.

† If 28.25 parts by weight of pure crystallized sulphate of copper ($\text{CuSO}_4 \cdot 5\text{OH}_2$) are dissolved in 71.75 parts by weight of distilled water, the resulting solution will have very nearly a specific gravity of 1.200 at 18° C.; and if 16.5 parts of the crystals are dissolved in 83.5 parts of water, the solution will have a density of 1.100 at 15° C.

For the zinc-sulphate solutions take 55.5 parts by weight of crystallized

rected Clark cell at 15°C. is very nearly $\cdot768$ to unity*; the Clark cell being taken as $1\cdot435$ at 15°C.

If, instead of taking the equidense solutions, we take zinc sulphate of specific gravity $1\cdot400$ and copper sulphate of $1\cdot100$ at 15°C. , and the same plates, the E.M.F. of the cell lies close to $1\cdot072$ volt, and the ratio of the E.M.F. to that of the corrected Clark is $\cdot747$.

If, however, instead of taking the electromotive force at once after the freshly electroplated copper pole is introduced into the cell, the cell is allowed to stand an hour or so, both the above values will be increased by about $\cdot003$ volt, provided always that no interdiffusion of solutions has taken place, and that the zinc retains a perfectly bright untarnished surface.

4. *Influence of Temperature on the Electromotive Force of the Normal Daniell Cell.*

The researches of experimenters who have studied the Clark cell have established the fact that its E.M.F. diminishes with rise of temperature. The exact value of the coefficient of variation seems to depend on the mode in which the cell is made up; and very careful examination has been made of the Daniell cell as above described, to see if its real E.M.F. is affected by temperature. Some writers have asserted that it is; but it is obvious that the variation of internal resistance must be eliminated entirely by the use of a null method of observation of the E.M.F. The condenser method is not a sufficiently accurate one to apply, and the potentiometer method is the most trustworthy.

Two U-tube cells were prepared with $1\cdot4$ sp. gr. zinc sulphate and $1\cdot1$ sp. gr. copper sulphate; and these cells were respectively immersed, the one in a water-bath at 20°C. and the other in melting ice at 0° . A pair of zinc and copper

zinc sulphate ($\text{ZnSO}_4\cdot7\text{OH}_2$) and dissolve in $44\cdot5$ parts of distilled water. The resulting solution will have at $20^{\circ}\cdot5\text{C.}$ a specific gravity of $1\cdot400$.

If 32 parts by weight of the crystals are dissolved in 68 of water at the same temperature, the solution will have a density of $1\cdot200$. These are useful densities.

* Lord Rayleigh assigns a value very close to $\cdot770$ for this ratio, and Dr. Alder Wright $\cdot765$. The figure in the text is derived from about 50 experiments of my own.

poles was likewise prepared; and when the solutions in the U-tubes had acquired the temperature of the baths, the E.M.F. of each cell was taken with the pair of zinc and copper plates, first in one and then in the other, with the following results:—

Plates in the warm cell at 20° C. E.M.F. of cell.	Plates in the cold cell at 0° C. E.M.F. of cell.
1·081	1·082
1·079	1·082
1·079	1·082
1·078	1·083

The experiments showed a very small but decided fall of E.M.F. as the cell is warmed, and at the rate of about 3 parts in 1000 for 20° C. This is only about $\frac{1}{50}$ of the variation of a Clark cell for the same range; and, practically, we may say that the E.M.F. of a Daniell cell is independent of temperature for such range as occurs in the natural temperature of the air in our climate*. This quality of the normal Daniell is very important, and goes very far towards helping it to sustain its position as a standard cell against its rival the Clark cell.

The various coefficients of temperature-change which have been found for Clark cells indicate that it is unsafe to rely upon any cell taken at random, whose history is unknown, having a coefficient of variation accurately the same as that of the cells whose coefficients have been actually determined.

It would be a great advantage if instrument-makers would construct these cells in the form suggested by Lord Rayleigh—enclosed in a test-tube and having gutta-percha-covered leading-in wires passing through a rubber stopper. The coefficients of variation would then be easily obtained for any cell, just as those of standard resistance-coils are ascertained and marked on each cell with the range of temperature over which it is applicable.

* The same result has been obtained by Mr. Preece in his experiments on the effects of temperature on the electromotive force and resistance of batteries, see 'Electrician,' March 3, 1883, vol. x. page 367. He gives a table showing the E.M.F. at various temperatures, obtained by the condenser method, and shows that there is no perceptible change in E.M.F. of a normal Daniell between 0° and 17°, and a slight fall subsequently of about 9 parts in 1000 between 17° and 28°, but a rise, however, after reaching 63°.

As a check on the foregoing experiments on the Daniell, the coefficient of variation of the standard Clark cell used was obtained. Two cells were taken, exactly equal at 18° C., and one of them was immersed for 48 hours in melting ice. The difference on the potentiometer-wire was then obtained between this cold cell and its fellow kept in water at 18° C. The potentiometer-readings were as follows:—

	i.	ii.	iii.
B, cell at 0° C. . .	961	971	974.4
A, cell at 18° C. . .	947	957	960
	14	14	14.4

From the experiment 1 the reading for the warm cell would have been $947 + \frac{14}{6} = 949\frac{1}{3}$ at 15° C. And hence the coefficient of variation is $\frac{14}{18}$ in $949\frac{1}{3}$, or .0819 per cent. in the neighbourhood of 15° C. Lord Rayleigh gives .082 as the value for his cells. It is obvious that Clark's cells cannot be used for standardizing galvanometers without a careful determination of their coefficients of variation.

The normal Daniell cell has a certain advantage in that when null methods for determining electromotive force are employed, its value is independent of temperature throughout a considerable range.

By the employment of the above two instruments—a potentiometer as described and a standard normal Daniell carefully constructed with all the precautions named—it is very easy to make very accurate measurements of strengths of currents of large magnitude. If a resistance is formed of such character that the current to be measured can be passed through it without much affecting its temperature, and the potential measured at the extremities of this resistance by means of a comparison on the potentiometer between it and that of the Daniell, this gives at once the current.

In order to construct a resistance whose value at the instant of passing the current can be accurately measured, the following device is adopted:—On a board are mounted a series of copper blocks *a, a, b, b*, like the connection-pieces on a resistance-box (see Pl. I. fig. 3). These can be connected by plugs *p, p*. At opposite sides and ends are main terminals as

M'; and the blocks *a*, *b* are connected by brass or German-silver wires wound in a spiral; similar coils of equal resistance connect each pair of opposite blocks. The number of coils must be an odd number.

If the plugs 1, 11, 3, 9, 5, 7 are put in, it is obvious all the coils $R_1, R_2 \dots R_7$ are a series, and their resistance can be measured: call it r . If, then, the other pegs are inserted, the coils are now in multiple arc or parallel, and if there are n coils the resistance is $\frac{r}{n^2}$. By this arrangement the resistance

in parallel can be inferred from the resistance in series; and although each coil has a very small resistance, yet by using a large number, not only is a very easily measurable resistance obtained when the coils are in series, but a resistance of large current-carrying capacity when they are put in parallel.

The mode of using it is as follows:—The pegs are all put in, and the current to be measured is sent through the coils in parallel. By means of the potentiometer and standard-cell, the difference of potential is found between the terminals M and M' when the current is passing. The alternate pegs are then removed, the current stopped, and the resistance taken of the coils in series, and the time noted which has elapsed between the measurement of potential and of the resistance. A few observations are taken of the resistance at intervals of time, by which to construct a curve of cooling; and by projecting back the curve, it is easy to ascertain very nearly the resistance at the time when the potential was measured. If small currents only are employed, it is not necessary to change the resistance; the current can be sent through it in series, and its resistance also measured in series. The writer has had constructed a large resistance of this kind at the Victoria Electric Lighting Station, for the measurement of dynamo currents up to 500 amperes. In this case the resistance consists of 36 wires of brass, No. 14 B.W.G., arranged in parallel, and of equal resistance. One of the wires can be disconnected and arranged instantly on a bridge to measure its resistance; and in this way, after measuring the potential at the ends of the 36 wires in parallel, with the current passing through them, one wire is disconnected, and its resistance taken immediately; from which observations it is possible to approximate very accurately to

the resistance of the whole wires in parallel. In order to afford data for constructing such resistances, experiments were made of passing various currents through coiled spirals of naked wires of different sizes and materials.

A large number of wires were prepared, of copper, brass, iron, German-silver, each 25 feet long, and of six sizes respectively, Nos. 10, 12, 14, 16, 18, 20, B.W.G., the diameters being given below. These wires were coiled into spirals round wooden rods about one inch diameter, and the turns of the wire well separated, so that each coil or spiral was about 18 inches long. Measured steady currents were sent through these for some hours, and so adjusted that after the temperature had become steady the wires were all at a temperature just about bearable by the hand, that is near 60°C . The currents respectively carried were as follows:—

Size of wire.	{	No. 10 B.W.G. ·134 inch diam.	12	14	16	18	20
		·109	·083	·065	·049	·035	
Currents carried in Amperes.							
German Silver ...	18·75	13·5	8·25	6	4·12	3	
Brass	30	18·75	15	9·75	7·5	5·25	
Iron	18	11·25	10·5	8·25	5·25	3·75	
Copper	49·5	38	26·25	20·25	15	9	

These currents passed through the above-described naked spirals bring the respective wires to about a temperature of 60°C ., when equilibrium is established; and for the purposes of measuring currents, not more than one third of the above currents should be used with wires of the size appended. Thus for 300 amperes, about 50 No. 10 B.W.G. wires will carry it without much sensible elevation of temperature; and by arranging 50 wires so that their resistance can be quickly measured in series, a resistance can be made suitable for measuring the potential at the ends of a known resistance.

By the use of this method it is easy to standardize any current or potential-galvanometer at any part of the scale, and

obtain the absolute or corrected value of the deflections; and these methods have been employed with success for some time in standardizing the working galvanometers used in the lamp factory of the Edison and Swan United Electric Light Company.

In conclusion may be given other results, collected from various authors, respecting the electromotive force of the Daniell cell.

Wiedemann (*Galvanismus*, vol. i. cap. iv., *Bestimmung der Electromotorischen Kraft*, p. 341) gives a description of Raoult's cell (see Raoult, *Ann. de Chim. et Phys.* [4] t. ii. p. 345, 1864), and states that Raoult finds that copper-foil has higher electromotive force than electro-deposited copper by about $\frac{1}{200}$; and attributes it to the oxides of copper contained in it. He also confirms the invariability of E.M.F. with temperature at least between 10° and 50° ; and states that with pure amalgamated zinc and electro-deposited copper, solution of saturated sulphate of copper and solution of sulphate of zinc, containing 1 part of crystals to 1 part of water = 1.35 sp. gr. at 20° , the electromotive force is 1.124 electromagnetic units. This, corrected by the Cambridge value of the B.A. unit, is 1.109 volt.

Sir W. Thomson (p. 245 of 'Papers on Electricity and Magnetism') finds the electrostatic measurements of a Daniell cell to be .00374. The nature of the solutions and electrodes are not given. Taking the value of v as 3×10^{10} , we have the E.M.F. as 1.12 B.A. volt, which, reduced to true volts, gives 1.105 volt.

Lord Rayleigh gives the ratio of a Daniell set up with amalgamated pure zinc, electro-deposited copper, and solutions of sulphate of zinc and copper, each of sp. gr. 1.1. Five observations of the ratio of this cell to that of a known Latimer-Clark cell at 16° C., taken as unity, gave .7702, .7710, .7705, .7698, .7702, mean .7703; and since the Clark is 1.435 volt, this gives the Daniell as 1.105 volt.

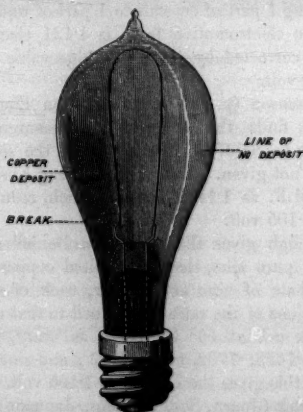
Latimer Clark (*Journ. Soc. Tel. Eng.*, January 1873) gives it as 1.11, which, reduced to true volts, is 1.095; and Dr. Alder Wright (*Proc. Phys. Soc. London*, vol. vi. p. 292) gives the value of a normal Daniell set up with solutions of the same molecular strength, preferably of strength m M SO_4 100 OH_2 ,

where m is near 2—that is with copper sulphate nearly saturated, and zinc sulphate of equivalent molecular strength, and pure amalgamated zinc plates and electro-deposited copper—as 1.114 B.A. unit, or 1.099 true volt.

None of these are very far from the value assigned to the standard cell described above, viz. 1.102 volt, with equidense solutions and metals as described.

XXII. *On Molecular Shadows in Incandescence Lamps.* By J. A. FLEMING, M.A., D.Sc. (Lond.), Fellow of St. John's College, Cambridge, Professor of Electrical Technology in University College, London*.

SINCE presenting a short note on the above subject in 1883 many further opportunities have occurred for observing the conditions under which molecular shadows are formed in incandescence carbon-filament lamps, and of correcting one or



two statements then made. It has been observed in an immense number of cases, that not only do incandescence lamps become

* Read June 27, 1885.

coated on the interior of the glass with a deposit of carbon, but that the envelope may have deposited upon it a metallic film, derived from the leading-in wires to which the carbon filament is clamped. In the Edison lamp the platinum wire only just passes through the glass, and is connected with copper wires broadened out into a clamp; the filament is gripped in these clamps and then electro-plated over with copper to effect a good junction. In the Swan lamp the platinum wires are joined directly on to the carbon. It occasionally happens that there is an unusual amount of resistance at the clamps, or that by excessive electromotive force more current is forced through the lamp and more heat generated everywhere. In this case the tendency will always be to increase and go on increasing the resistance, and, therefore, the temperature at the point of highest resistance. Suppose this occurs on the clamp or on the leading-in wire, then experience shows that the metal is volatilized and deposited as a film on the glass. This metallic deposit is not uniform; it is thickest nearest the base of the lamp and gradually thins away up to the crown, and at a certain height is thin enough to transmit light. It is not very uncommon to obtain Edison lamps with this copper deposit. The colour of transparent copper is a fine sage green inclining to blue; compared with the colour of gold leaf seen by transmitted light, it very closely resembles it. On one occasion the writer obtained a lamp with a silvery metallic film deposited over it on the inside of the glass. From the outside it had a mirror-like lustre; on breaking the lamp this film was seen to be brownish and not brushed off by the finger, but it could be removed by scratching. It was not removed by holding in the oxidizing flame of a Bunsen burner; therefore it was not carbon. It was not removed by nitric acid; but on boiling a fragment of the glass, covered with this metallic deposit, in nitro-hydrochloric acid the film disappeared. It was therefore probably platinum. The film was transparent, permitting objects to be seen through it, and transmitted brownish light.

Now, under certain circumstances, a line of no deposit is formed on the surface in the plane of the filament, which is, as it were, the shadow of one side of the loop. This indicates that the process of molecular scattering, which is going on at

some spot on one or other clamp, is not a mere evaporation or volatilization of the metal, but a projection of molecules in straight lines in every direction. The trajectory of the molecules will be interfered with in some directions by the carbon filament; and hence result lines and places of no deposit which are molecular shadows of the loop. On every other part of the glass the molecules will impinge and adhere, forming a metallic coating. From the facts that the free paths of the molecules differ in length, and that the clamp is much nearer to the neck of the lamp than to the crown, it follows that a much larger proportion of the scattered molecules strike the glass near the neck, and the thickness of the deposit is therefore a measure of the proportion of molecules which have a free path, equal to the distance of that part of the envelope from the scattering point. Curiously enough the line of no deposit, or shadow of the loop, is not always seen in copper-deposited lamps. This may be because the scattering is going on from both clamps, and therefore the shadow on one side is covered up by the shower from the clamp on that side. It has been noticed in one or two cases that small tufts of carbon are seen on one clamp, and that when a well-defined shadow exists on that side, this seems to indicate that the shower of copper molecules has been partly stopped by the opposite clamp, which has therefore acted like a target and become encrusted with a proportion of the molecules shot at it.

With respect, next, to carbon deposits. Every one knows the appearance of a lamp after it has been burning for some time or overburnt: it is clouded with smoke-like deposit. In nearly all cases of copper deposit the molecular shadow exists, but it is not nearly so often seen in the case of carbon deposits. After many observations it was found that the molecular shadow of the filament, or line of no deposit, could be formed by suddenly raising the filament to a very high temperature, as for instance by placing a 50-volt lamp for an instant on a 100-volt circuit; whereas when the deposit of carbon takes place slowly, and as it were in the natural way, the lamp exhibits only a general smokiness but no line. Again, it has been found that when a carbon filament is cut sharply through at one point, caused by excessive temperature at one spot,

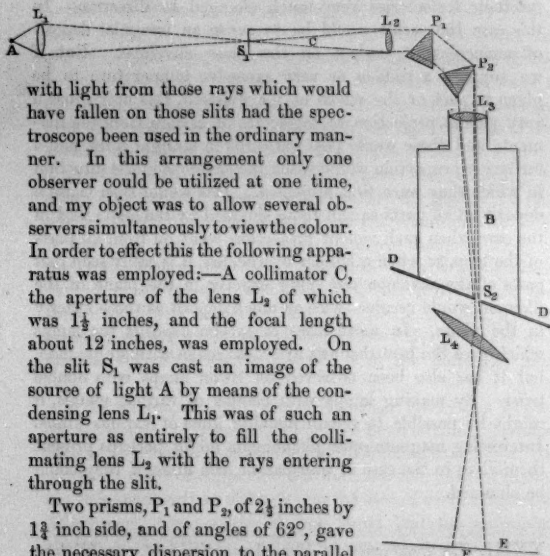
there is very frequently a sharply marked line of shadow of the loop on the side of the envelope farthest removed from the fracture. These facts seem to indicate that in normal use, when the lamp is not being pressed beyond the electromotive force at which it was intended to be used, there is a general evaporation of carbon going on from all parts of the loop, and these molecules, being projected with no abnormal velocity, probably collide with molecules of residual air a large number of times before they reach the walls of the envelope, and thus get their trajectories very much changed in direction. In this case the result would be to cause an irregular deposit of molecules of carbon on the glass envelope. But if we suppose a sudden or very excessive temperature to be given to part or the whole of the filament, this may cause a very violent projection of molecules of carbon from the filament; and these would pass outwards in straight lines, and a far larger proportion would reach the envelope in the direction in which they were first projected. This would then cause a deposit on all parts except those shielded by the loop; and in the case when such violent projection went on from all parts of the loop, as when a lamp is overheated, it is easily seen that parts of the envelope not lying exactly in the plane of the filament would receive twice as much deposit as those exactly in the plane. In most cases of carbon deposits the lamps which give the best shadows are those made with single loop, but it has also been observed on Swan lamps with double twist. By making lamps with clamps of various metals, it might be possible to obtain metallic films of various kinds. Interesting magneto-optic phenomena might perhaps present themselves in the case of transparent iron films, if they could be obtained.

XXIII. *The Production of Monochromatic Light, or a Mixture of Colours, on the Screen.* By Capt. W. DE W. ABNEY, R.E., F.R.S.*

FOR some time past I have been making experiments on the illuminating power of various sources of light, and it became necessary in some investigations to produce fair-

* Read June 27, 1885.

sized patches of different monochromatic, and combinations of monochromatic lights, upon a screen, all other colours being absent. To obtain this result I had recourse to a modification of Clerk-Maxwell's arrangement, as used for colour-mixing. By his plan slits were inserted at different parts of the spectrum as formed by an ordinary spectroscop. These slits were then illuminated by light reflected from a white screen, and the prism was viewed through the slit of what is ordinarily the collimator. The prism was then seen to be coloured



with light from those rays which would have fallen on those slits had the spectroscop been used in the ordinary manner. In this arrangement only one observer could be utilized at one time, and my object was to allow several observers simultaneously to view the colour. In order to effect this the following apparatus was employed:—A collimator C, the aperture of the lens L_2 of which was $1\frac{1}{2}$ inches, and the focal length about 12 inches, was employed. On the slit S_1 was cast an image of the source of light A by means of the condensing lens L_1 . This was of such an aperture as entirely to fill the collimating lens L_2 with the rays entering through the slit.

Two prisms, P_1 and P_2 , of $2\frac{1}{2}$ inches by $1\frac{1}{4}$ inch side, and of angles of 62° , gave the necessary dispersion to the parallel beam, from the collimator. The rays were brought to a focus by means of the lens L_3 , of about 14 inches focal length, on to the screen belonging to a camera, B. This screen was placed at an angle with the axis of the lens L_3 as shown, so that a fair focus of every visible ray was obtained upon it. (It may be worth mentioning that a hair placed across the slit or a

little particle of dust is a good means of obtaining a focus when Fraunhofer or bright lines are not observable. The black streak produced by it should be sharp along the whole of the spectrum.) A card, D D, with a slit S_2 or slits cut in it, replaced the ordinary dark slide, and, by moving it along the spectrum, any colour or colours can be allowed to pass. Before using the apparatus the whole of the spectrum was caused to fall on a convex lens, L_4 , of about 24 inches focal length and about 5 inches diameter. This collected the dispersed beam of light, giving an enlarged image, F, of one surface of one of the prisms on a screen, E. By placing this lens at an angle with the axis of the lens L_3 , the blue and red fringes can be made to disappear almost entirely, and a practically white patch of light is seen on the screen E.

I may say that the lenses used are white flint of medium density and almost colourless, even in great thickness.

When the adjustments are complete, as the slit is moved along the spectrum every patch of colour or colours will successively occupy the same position on the screen and have the same area very nearly. We thus can have patches of monochromatic light of any colour or combinations of any colours, all other colour being absent.

I have also obtained the same results by substituting mirrors for all the lenses and a reflecting-grating for the prisms; but I do not see any particular advantage in this plan, as the white light is more tinged (with the colour of the metal) than when prisms and lenses of white glass are employed.

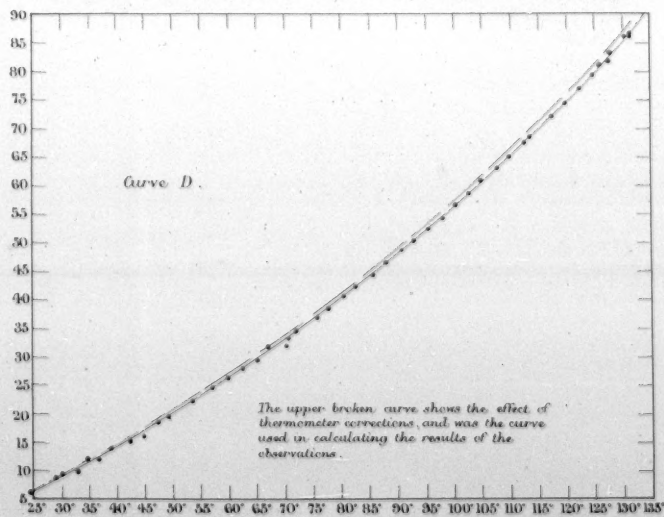
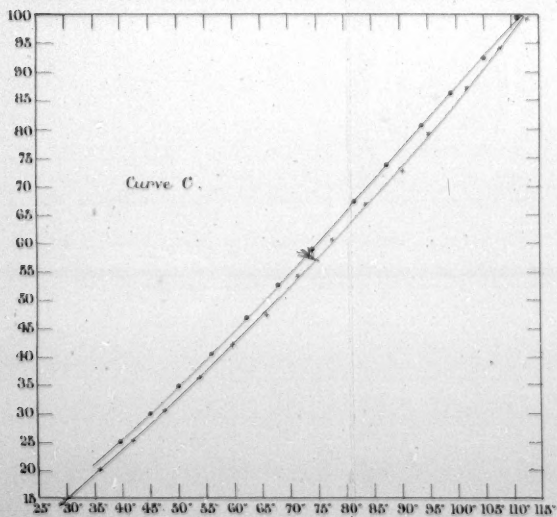
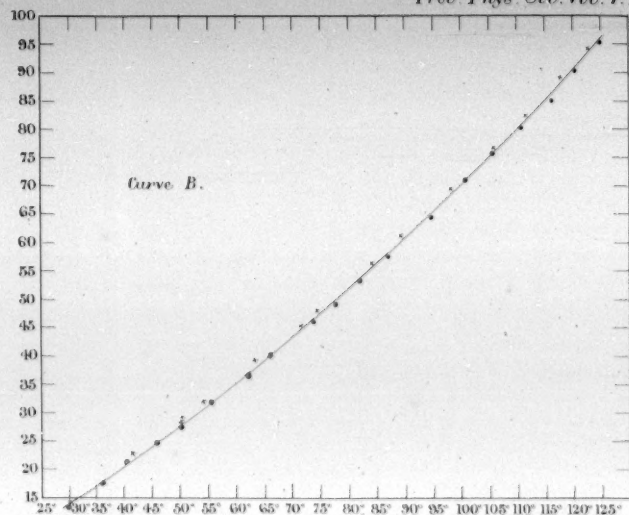
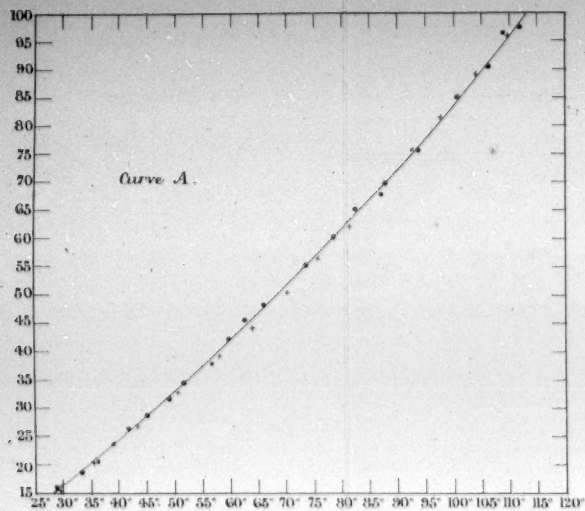
When the source of light is the arc light, if an image of the crater of the positive pole be thrown on the slit S_1 of the collimator, the intensity of the light is such (when the slit is fairly open) that the patch of nearly pure light may be made 1 foot square, and yet be sufficiently brilliant to be seen by a fairly large audience, and for an ordinary lecture-room it is very effective. The mixture of colours to imitate any colour in the spectrum may be shown by placing a narrow slit in a small card in the colour required to be imitated, and fixing in front of it and in contact with it a portion of a cylindrical lens, the axis of the cylinder being of course parallel to the slit. This throws the image of the particular colour to be observed at one side of the image that would be

obtained were the cylindrical lens absent, and any amount of deviation can be given the patch by using the different parts of the lens. Thus the greater the deviation required the nearer to the margin of the lens the part of it employed should be. In other words, the cylindrical lens acts as a series of prisms of varying angles. To obtain mixtures of colours to correspond to the deviated patch, movable slits, and capable of being narrowed or widened, are placed on each side of the fixed slit. By this plan two patches of light of equal size and equal intensity can be readily produced. When measurements are to be obtained, scales are attached to the various slits, by which any part of the spectrum can be identified; and the widths of the slits are measured by a gauge.

Since this apparatus was described I have referred to a paper by Helmholtz, which appeared in Poggendorff's *Annalen* in 1855, in which one of the methods he used for the combination of coloured light to produce white light is described. The general principle he adopted is the same as that described above; but in several important details the latter differs considerably from Helmholtz's apparatus. For instance, the apparatus now described is suitable for the comparison of colour-mixtures with monochromatic light of any colour and for their exhibition on the screen for lecture-purposes, and an illumination is secured which is very largely in excess of that usually obtained. In a paper read before the Physical Society (Phil. Mag. June 1885) Lord Rayleigh shows how a monochromatic image of an external object may be seen by placing a concave lens immediately behind the slit of the spectroscope of such a power as to throw an image of that object on the prism. I have found that by altering the distances apart of the collimating lenses and viewing lens, a monochromatic image of the sun may be thrown on the screen. If such an image be coloured with the light of the blue or violet hydrogen lines, it should be possible to photograph the solar prominences *en bloc*. I may mention that the apparatus as described was employed in two Cantor Lectures at the Society of Arts in the beginning of April, but without describing it in detail.

I am at present engaged in using this apparatus for investigating some phenomena existing in colour-blindness, and obtaining curves of illumination of different lights, and also





in some photographic researches. These results are not yet ripe for publication ; but I have thought it might be well to publish the method employed, as it is one of great convenience, and very easily carried out by any one who has a spectroscope and a photographic camera.

XXIV. *On the Thermoelectric Position of Carbon.*

By JOHN BUCHANAN *.

[Plate II.]

In the very interesting paper read by Dr. J. A. Fleming before this Society on March 14, 1885, whilst discussing the question of the life of incandescent lamps, the author remarks that quite a large proportion of the carbon filaments are found to break at the negative end, that is near the point where the current passes from carbon to platinum. It seemed to me not improbable that the negative end of the carbon might be subjected to a heating action due to the operation of the "Peltier effect." The local excess of temperature thus produced might possibly account for the observed result. In any case the determination of the position of the carbon line on the thermoelectric diagram was of interest. I am not aware that this has been done before.

Result.—For the specimen of carbon I have tried, the Thomson effect is of the same sign as in copper, and about twice the value for that metal given by Prof. Tait, as quoted in Everett's "Units and Physical Constants," § 186. The thermoelectric power of carbon is given by my experiments,

$$e_c = -390 - 1.87t \text{ C.G.S. units,} \quad \dots (1)$$

where t denotes, as usual, the mean temperature of the junctions. The neutral point of carbon with lead is therefore about -209°C .

Thermo-couple of Platinum-Carbon.—As I have mentioned, the series of experiments described below were undertaken with the object of finding the magnitude of the Peltier effect in incandescent lamps. I therefore examined specially the behaviour of a thermoelectric couple of platinum-carbon.

* Read June 27, 1885.

The platinum was a piece of ordinary commercial wire procured from Messrs. Johnson and Matthey. The carbon was in the form of rods 12 millim. diam., such as are made for use in arc lamps. The direction of the current in such a couple is from platinum to carbon across the hot joint. The thermoelectric power at the mean temperature t is

$$e_{pt-c} = +566 + 3.94t \text{ C.G.S. units.} \quad (2)$$

Due, therefore, to the "Peltier effect," heat will be generated at the negative end of the carbon filament of an incandescent lamp when the current is passing. We can now calculate what proportion the heat thus generated locally bears to the whole quantity of heat generated in the filament.

If T denote the neutral point of a thermo-couple, then the Peltier effect for the absolute temperature θ at one joint is given by

$$\pi = (K_1 - K_2)(T - \theta)\theta, \quad (3)$$

where K_1 and K_2 are the numbers that express for each material the ratio that the Thomson effect bears to the absolute temperature. (See Tait's 'Heat,' §415.)

Hence for a platinum-carbon couple,

$$\pi = 3.94(144 + \theta)\theta. \quad (4)$$

As a specific example, take the case of a 100-volt lamp, with the filament at a temperature of 2400°C . We get the ratio

$$\begin{aligned} \frac{\text{Heat due to Peltier effect}}{\text{Whole heat in filament}} &= \frac{3.94 \times 2673(144 + 2400)}{100 \times 10^8}, \\ &= \frac{2.7 \times 10^7}{10^{10}}, \\ &= .0027. \end{aligned}$$

The local heating at the negative end of the filament is therefore only about one quarter per cent. of the whole heat. This is too small to have any appreciable influence on the life of the lamp. It does not appear therefore that the Peltier effect accounts for the giving way of the carbon-filament at one end rather than at the other. In the course of these experiments one phenomenon was forced on my attention, the details of which are given below. It seems that con-

tinuous heating alters the thermoelectric properties of carbon. This alteration must be the result of molecular change in the material. That this has any bearing on the destruction of the filament that ensues, by keeping it for a long time at a high temperature, my experiments are not extended enough to decide.

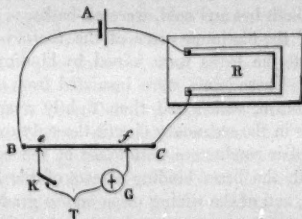
Experimental Details.

The E.M.F. at various temperatures was determined by the null method known as Poggendorff's.

The primary circuit contained a constant cell A, a graduated wire BC, and a box of resistance-coils R.

On the graduated wire were placed sliding pieces *e* and *f*, which served as the terminals of the secondary circuit. In the secondary circuit were a galvanometer G, and the thermo couple T whose E.M.F. was to be measured, and a key K.

Fig. 1.



The arrangement of the apparatus used in the platinum-carbon experiments has now to be described.

Three rods of carbon of 1.2 centim. diameter, such as are manufactured for use in arc-lamps, were taken. One of these was thinned down into a conical form at the end destined to go into the hot bath. This end, and that of another rod intended to go into the cold bath, were electroplated with a thin shell of copper. To this copper shell were soldered the platinum wires, whose behaviour with the carbon I wished to examine. These two rods were then placed vertically and clamped in position with coppered ends dipping into their respective baths. The upper ends of these carbon rods were cut off square and the third rod laid horizontally, so as to

form with the vertical rods the third side of a vertical rectangle.

To ensure good contact one end of this third rod was bound by a number of turns of clean iron wire to the upper end of the rod whose lower end dipped into the cold bath. The other end of this third rod rested on the top of the rod the lower end of which dipped into the hot bath. The surfaces of contact were here scraped flat and kept pressed together by a weight of 3 or 4 pounds.

The platinum wires that formed the other element of the thermo-couple were of ordinary commercial material, as was mentioned above.

The galvanometer used in the secondary circuit is a very delicate mirror instrument of about $\frac{1}{4}$ ohm resistance.

The cell used in the primary circuit as a standard was a Daniell. The graduated wire (BC in fig. 1) was of German silver.

The baths, both hot and cold, were oil-baths.

The ends of the platinum wires of the thermo-couple that were exterior to the baths were joined by binding-screws to copper wires. These joints were insulated from one another by several layers of calico, and then tightly wrapped up together. Thus in the secondary circuit the only exposed contacts of dissimilar conductors were those of the copper wires joined up with the brass binding-screws of the key, of the galvanometer, and of the sliding piece on the graduated wire. During the experiment, care was taken to screen these exposed contacts as much as possible from external radiation. Moreover, all the apparatus was set up the night before, so that on the days of the experiments no metal joints whatever in the secondary circuit were touched by hand.

Some preliminary experiments I had made showed the necessity for adopting the arrangements just described. The results of these preliminary trials exhibited very curious irregularities, not altogether got rid of even by those latest arrangements. By using the precautions detailed above, I ensured, however, that the place of any disturbances that might come into play should be located with certainty in the thermo-joints under examination.

The proper amount of resistance to be inserted in the

primary circuit having been determined beforehand, the primary circuit was closed. The current was allowed to flow during the whole time the experiment lasted. The temperature of the hot bath was raised very slowly up to the highest point reached; then as slowly lowered again. Frequent readings of the thermometers in the hot and cold baths were taken. Simultaneously were noted the positions of the movable sliders on the graduated wire, for which no deflection was shown by the galvanometer on depressing the key in the secondary circuit.

When a temperature reading was intended to be taken, the temperature was kept as steady as possible for $\frac{1}{2}$ to $\frac{1}{3}$ a minute, accompanied with constant stirring of the hot bath. In this way a series of corresponding readings of temperature and of distances on the graduated wire were obtained. The results were plotted out, taking centimetres of the graduated wire as ordinates, and temperatures of the hot bath as abscissæ. A regular curve was then drawn amongst the points thus obtained, so as to give as nearly as could be judged the mean of the observations. The curves A, B, C (Plate II.) were obtained in this way. The scales for these curves are not quite the same. Details are given below. The curves A and B were obtained on successive days.

The want of regularity in the individual readings, after all the precautions taken, I considered could only be due to the properties of the carbon. Partly to satisfy myself that no essential precaution had been overlooked, however, and partly to test an idea I formed as to the cause of the irregularities, the apparatus was set up again about a month later. Curve C gives the results of the observations. The heating of the hot joint gave the curve to the left hand; the subsequent cooling gave the right-hand curve. These curves have not a point of intersection at their upper ends shown in the diagram (Plate II.), because the temperature was intentionally raised to a point higher than that for which a reading on the graduated wire could be obtained, before cooling down was begun.

Now, as $4\frac{1}{2}$ hours were given to the observations, the heating and cooling took place very slowly indeed. And as a stirrer was kept almost continuously in operation in the hot bath, the lagging of the thermometer behind the thermo-

joint must have been very small. I believe that the difference between the curves obtained by heating and by cooling is partly due to change taking place in the properties of the carbon as the temperature changes. Bearing on this there is an observation I made. Having calculated the neutral point of a carbon-iron couple as about $+208^{\circ}\text{C}$., a glycerine bath was used to verify this result. Having raised the temperature to about 290°C . and cooled again to 220°C . several times, I found the E.M.F. of the carbon-iron couple rose gradually, until at the time of stopping the experiment the maximum E.M.F. had increased by more than 50 per cent. As is well known, a thermo-couple reaches a maximum E.M.F. when the temperature of one joint is that of the "neutral point." It was this maximum that was found to increase with the time during which heat was applied.

The conclusion is, that the thermoelectric properties of carbon alter with temperature. Principally for this reason I did not deem it of utility to begin a new series of observations, and by "loading" the hot joint with metal try to decrease the irregularities in the readings like those in curves A and B. Moreover, the results calculated from these curves agree as well as could be expected. Here are the details:—

Curve A.

Assuming the equation to the curve as

$$y = K(x-a)\left(T - \frac{x+a}{2}\right),$$

I find that $T = -147^{\circ}\text{C}$., $K = 0.00470$

satisfy best the observations as represented by the curve A.

x = temperature of hot bath,

a = " cold bath,

T = neutral point.

The agreement of calculated and observed values of y is exhibited by Table I.; these values, it will be recollected, represent centimetres of the bridge-wire. To calculate the E.M.F. of a carbon-platinum thermo-couple as given by curve A, we have the following data:—

Resistance of primary circuit = 303.6 ohms.

Observed E.M.F. of Daniell used = 1.03 volt.

Resistance of 1 centim. of the graduated wire = $.00246$ ohm.

Denoting the E.M.F. required by E_A we get

$$E_A = 0.00470 \times 1.03 \times 10^8 \frac{0.00246}{303.6} (x-a) \left(147 + \frac{x+a}{2}\right),$$

$$\therefore E_A = 3.92(x-a) \left(147 + \frac{x+a}{2}\right) \text{ C.G.S. units.}$$

Hence the thermoelectric power of the couple is

$$e_A = +577 + 3.92 \times t \text{ C.G.S. units,}$$

t denoting the mean temperature of the joints. The direction of the current would be from Pt to C across the hot joint.

Curve B.

Taking an equation to the curve of the same form as that just used, $T = -140^\circ \text{ C.}$ and $K = 0.00406$ best satisfy observation.

TABLE I.

$x.$	$a.$	$y.$	
		Calculated.	Observed.
30°	9.75	15.9	15.8
40	9.82	24.4	24.2
50	9.85	33.4	33.4
60	"	42.9	43.1
70	"	52.9	53.0
80	9.9	63.2	63.4
90	"	74.2	74.1
100	"	85.5	85.1
110	"	97.4	97.7

TABLE II.

$x.$	$a.$	$y.$	
		Calculated.	Observed.
30°	9.6	13.5	13.2
40	9.65	20.3	20.3
50	9.70	27.8	27.8
60	9.75	35.7	35.7
70	"	43.8	44.0
80	"	52.6	52.7
90	9.80	61.9	61.9
100	"	71.7	71.4
110	9.85	81.8	82.4
120	"	92.6	91.6

Table II. exhibits the agreement of the calculated and observed values of y when these constants are inserted in the

equation to the curve. To calculate E.M.F. (E_B), the data are the same as were used in the calculation of E_A , except that the resistance in the primary circuit was 259.6 ohms.

$$\therefore E_B = 0.00406 \times 1.03 \times 10^8 \times \frac{0.00246}{259.6} (x-a) \left(140 + \frac{x+a}{2} \right) \\ = 3.96(x-a) \left(140 + \frac{x+a}{2} \right) \text{ C.G.S. units.}$$

Hence for the thermoelectric power (e_B) at the mean temperature t , we get

$$e_B = +555 + 3.96t \text{ C.G.S. units.}$$

The mean of e_A and e_B being denoted by e_{pt-c} , we get

$$e_{pt-c} = +566 + 3.94t \text{ C.G.S. units,}$$

the result quoted in equation (2) above.

Thermoelectric Power of Carbon.

In order to find the position of carbon on the thermoelectric diagram, I determined the line on this diagram of the platinum employed in the above experiments.

A strip of commercial lead, about 80 centims. long, had the platinum wires soldered to the ends. The general arrangement of the apparatus and the precautions used were similar to what have already been described above. Curve D shows the results of the observations; the ordinates and abscissæ being the same as in the curves A, B, C. It will be noticed that the observations lie well on the mean curve, thus showing that the arrangements were not defective in any essential point.

Assuming

$$y = K(x-a) \left(T - \frac{x+a}{2} \right)$$

as the equation to the curve, as above, I find that $T = -85^\circ \text{C.}$ and $K = 0.004815$ give results agreeing best with observation. (See Table III.)

TABLE III.

x.	a.	y.	
		Calculated.	Observed.
30°	13.1	8.67	9.05
50	"	20.7	20.6
70	"	34.6	34.7
90	"	50.5	50.8
110	"	68.3	68.2
130	"	88.1	88.0

The data for calculation are :—

Resistance of primary circuit = 1001 ohms.
 Mean observed E.M.F. of the Grove cell
 employed } = 1.75 volt.

Hence the E.M.F. of the lead-platinum couple and the thermoelectric power of platinum are, respectively,

$$E = .004815 \times 1.75 \times 10^8 \frac{.00246}{1001} (x-a) \left(85 + \frac{x+a}{2} \right);$$

$$\therefore E = 2.07(x-a) \left(85 + \frac{x+a}{2} \right) \text{ C.G.S. units,}$$

$$\therefore e_{pt} = +176 + 2.07t \text{ C.G.S. units.}$$

The current would be from Pt to Pb across the hot junction.

Using this value for e_{pt} , we find for the thermoelectric power of carbon

$$e_c = -390 - 1.87t \text{ C.G.S. units.}$$

This is the result quoted in equation (1) above.

In the table of thermoelectric powers given by Everett ('Units and Physical Constants,' § 186) there are given

$$\text{for Zn, } e_{zn} = -234 - 2.40t,$$

$$,, \text{ Ag, } e_{ag} = -214 - 1.50t.$$

The value for e_c shows that the place of carbon in the table is between zinc and silver. In the thermoelectric diagram it is easy to see that for temperatures below 50°C. , the line for carbon lies below that for cadmium, taking the value for e_{cd} from Everett's table (*loc. cit.*).

Concluding Remarks.

The disintegration preceding complete destruction of the carbon filament in an incandescent lamp, described by Mr. W. H. Preece, F.R.S., in a paper read before the Royal Society on March 26, 1885, is doubtless the result of molecular changes in the structure of the filament produced by the continued maintaining of the filament at an extremely high temperature. In my experiments, keeping carbon at a moderately high temperature altered the molecular condition of this material, the alteration being manifested as a change in the thermoelectric power of the carbon.

Physical Laboratory, University College,
 London, June 1885.

XXV. *On the Specific Refraction and Dispersion of Light by the Alums.* By J. H. GLADSTONE, Ph.D., F.R.S.*

In the *Comptes Rendus* of November 17, 1884, there appears a paper by M. Charles Soret, in which he gives the refractive indices of a large number of crystallized alums. They were determined by the method of total reflection, and are measured for the lines of the solar spectrum *a*, B, C, D, E, *b*, F, and G. As at the same time M. Soret gives the specific gravity of the specimens examined, it is easy to calculate out the specific refraction and dispersion of these crystals, and it occurred to me that the observations might serve three purposes. They might test the truth of certain physical laws; they might arbitrate between myself and Kanonnikoff as to the refraction-equivalent of certain metals; and they would give values for indium and gallium, which have not hitherto been obtained. Through the kindness of M. Soret I have received his full paper†, with some further information, and observations on three additional alums not yet published, viz. the Rubidium Gallium, Rubidium Indium, and Cæsium Chromium alums.

In order to bring M. Soret's figures into comparison with my own, which are always based on the line A of the spectrum, it was necessary to reduce his figures accordingly. This was easily done, as the line A falls short of *a* by almost exactly the same amount as that one falls short of B. The following table gives the specific gravity of the selected crystals; the refractive indices (μ) for A calculated to the fourth place of decimals, and also those for G; the specific refraction for A, i. e. the index -1 divided by the density, $\left(\frac{\mu_A - 1}{d}\right)$; the specific dispersion A to G, i. e. the difference between the specific refraction of these two lines, $\left(\frac{\mu_G - \mu_A}{d}\right)$; and the refraction-equivalent for A, i. e. the specific refraction multiplied by the

* Read June 27, 1885.

† "Recherches sur la Refraction et la Dispersion des Aluns cristallisés," *Archives des Sciences Physiques et Naturelles*, vol. xii. p. 553, and vol. xiii. p. 5.

atomic weight, $\left(P \frac{\mu_A - 1}{d}\right)$. The general formula of the alums is $R_2, Al_2, 4SO_4, 24H_2O$; the aluminium being replaceable by indium, gallium, chromium, and iron.

Alum.	Specific gravity.	μ_A .	μ_G .	$\frac{\mu_A - 1}{d}$.	$\frac{\mu_G - \mu_A}{d}$.	$P \frac{\mu_A - 1}{d}$.
Ammonium Aluminium..	1.631	1.4542	1.4692	.2784	.0093	252.23
Sodium " ...	1.667	1.4342	1.4480	.2604	.0063	238.52
Methylamine " ...	1.568	1.4496	1.4636	.2867	.0090	267.68
Potassium " ...	1.735	1.4516	1.4661	.2603	.0083	246.81
Rubidium " ...	1.852	1.4513	1.4662	.2437	.0080	253.69
Cæsium " ...	1.961	1.4536	1.4682	.2313	.0074	262.31
Thallium " ...	2.257	1.4914	1.5108	.2177	.0086	278.34
Ammonium Chromium...	1.719	1.4781	1.4959	.2781	.0104	265.94
Potassium " ...	1.817	1.4754	1.4931	.2616	.0098	261.22
Rubidium " ...	1.946	1.4756	1.4932	.2444	.0090	266.69
Cæsium " ...	2.043	1.4753	1.4928	.2326	.0086	275.50
Thallium " ...	2.386	1.5158	1.5381	.2161	.0092	286.90
Ammonium Iron	1.713	1.4783	1.4998	.2792	.0126	269.15
Potassium "	1.806	1.4757	1.4960	.2634	.0112	265.03
Rubidium "	1.916	1.4763	1.4970	.2486	.0108	273.21
Cæsium "	2.061	1.4772	1.4984	.2315	.0103	275.95
Thallium "	2.385	1.5155	1.5411	.2161	.0108	288.85
Rubidium Indium.....	2.065	1.4586	1.4740	.2221	.0074	269.63
Rubidium Gallium	1.962	1.4606	1.4758	.2348	.0077	264.43

Messrs. Topsoe and Christiansen* have also published the refractive indices of three alums for the lines C, D, and F; and as they have given the specific gravity, their observations are available for the same purpose. The index for A has been calculated from these data.

Alum.	μ_A .	μ_C .	$\frac{\mu_A - 1}{d}$.	$P \frac{\mu_A - 1}{d}$.
Potassium Iron Sulphate	1.4751	1.4783	.2597	261.37
Ammonium Iron Sulphate	1.4789	1.4821	.2786	268.56
Potassium Aluminium Selenate	1.4748	1.4773	.2400	272.98

How far do these figures support the former conclusion, that a salt has the same specific refraction whether in a solid form or in solution?

* "Krystallografisk-optisk Undersøgelser," *Det K. Danske Videnskabernes Selskabs Skrifter*, 1873, p. 622.

For this purpose determinations were made of the refraction for A in aqueous solutions of the first two alums in M. Soret's list. Deducting the amount due to the solvent, the following values were arrived at:—

	Dissolved.	Crystallized.
Ammonium Aluminium alum . . .	·2780	·2784
Sodium Aluminium alum . . .	·2613	·2604

These numbers are as close as two different specimens of the same salt, even if they were in the same condition, are likely to be. It did not seem worth while to obtain additional proof of a law which has already been established by many cases; not only by my previous observations, but also by those of Topsoe, Bedson, and Kanonnikoff.

Do these figures confirm the law that the refraction-equivalent of a compound body is the sum of the refraction-equivalents of its components?

The alums may be regarded as a compound of the sulphates of two metals of different kinds with 24 molecules of water. Now water in the uncombined state has the refraction-equivalent of 5·926; and in regard to the sulphates we have the following data:—

Ammonium sulphate, $(\text{NH}_4)_2\text{SO}_4$, 39·3. The mean of five determinations made by myself gives 39·35*: Kanonnikoff gives 39·27†.

Sodium sulphate, Na_2SO_4 , 26·6. Kanonnikoff gives 26·02.

Methylamine sulphate, $(\text{NCH}_3)_2\text{SO}_4$, 54·5; *i. e.* ammonium sulphate with the addition of 2CH_3 , or 15·2.

Potassium sulphate, K_2SO_4 , 32·4. The mean of five determinations gave me 32·6: Kanonnikoff gives 32·22. It may be estimated at 32·3 from Topsoe and Christiansen's determinations of the crystallized salt.

Rubidium sulphate, Rb_2SO_4 , 41·3. This is Kanonnikoff's determination, which I prefer to my own made in 1869.

Cæsium sulphate, Cs_2SO_4 , 55·3. Also Kanonnikoff's.

Being not content with my old observations on the sulphates of the trivalent metals, I have recently determined them afresh, with the following results:—

Aluminium sulphate, $\text{Al}_2(\text{SO}_4)_3$, 70·5.

* Four of these are given in my paper in the Phil. Trans. 1869, p. 9.

† Journal of the Russian Physico-Chemical Society, 1884, p. 119.

Chromium sulphate, $\text{Cr}_2(\text{SO}_4)_3$, 82.5; the result of old and new observations.

Ferric sulphate, $\text{Fe}_2(\text{SO}_4)_3$, 89.1; the mean of old and new observations.

It is evident that on adding together

Ammonium sulphate . . .	39.3
Aluminium sulphate . . .	70.5
24 Water	142.2

We obtain 252.0

In this way the second column of the following table has been calculated:—

Substance.	Refraction-equivalent.		
	Calculated.	Observed, and reduced to A.	
		Soret.	Topsee and Christiansen.
Ammonium Aluminium alum.....	252.0	252.2	
Sodium " "	239.3	238.5	
Methylamine " "	267.2	267.7	
Potassium " "	245.1	246.8	
Rubidium " "	254.0	253.7	
Cæsium " "	268.0	262.3	
Ammonium Chromium alum	264.0	265.9	
Potassium " "	257.1	261.2	
Rubidium " "	266.0	266.7	
Cæsium " "	280.0	275.5	
Ammonium Iron alum.....	270.6	269.1	268.6
Potassium " "	263.7	265.0	261.4
Rubidium " "	272.6	273.2	
Cæsium " "	286.6	276.0	

The agreement between the results calculated and found is as near as might be expected, except in the case of cæsium. It confirms the general law, as the variations only in one instance amount to as much as 1 per cent., and are sometimes plus and sometimes minus. There is little doubt that the cæsium, in one set of observations or the other, was impure: an old determination of mine, from the chloride, would give figures lower than Soret's.

Do these data afford us the means of determining the refraction-equivalents of the elements with more exactness than heretofore? It is evident that in the series of aluminium

alums the metal having the smallest refraction-equivalent is sodium, and that the rest follow in the order—potassium, ammonium, rubidium, methylamine, caesium, and thallium; the same order is preserved in the chromium and iron alums as far as they extend. This agrees with the order previously determined both by myself and Kanonnikoff; but when we look more closely into the matter it is evident that the figures are not very exact. Thus, in the aluminium series the difference between 2NH_4 and 2K is 5.42; but in the chromium series it is only 4.72, and in the iron series 4.12, or 7.19 according to Topsoe. The values of potassium deduced from this would vary considerably: assuming the value of NH_4 to be 11.1, it might be either 8.4, 8.7, 9.0, or 7.5; which are wider differences than between my old estimation, 8.1, and Kanonnikoff's recent independent determination, 7.75. This will not be wondered at when it is remembered that all experimental errors are accumulated on these residual numbers. The optical determinations of Soret are so uniform and exact as to inspire the greatest confidence; but he is not satisfied with the specific gravities. Errors may also arise from impurity of the salt, or want of homogeneity in the crystal. An aqueous solution is probably a more uniform substance than a hydrated crystal, and better fitted for the purpose of determining optical equivalents.

On this account any determination of the refraction of indium and gallium made from these alums must be open to considerable question: the rubidium compounds of these metals, however, are believed by Soret to be fairly pure salts, and we can compare them with three other alums of rubidium. It would appear from the first table that indium has the value of aluminium +8.0, or of chromium +1.5, or of iron -1.8; and, similarly, that gallium has the value of aluminium +5.4, or of chromium -1.1, or of iron -4.4. Now aluminium sulphate has already been estimated at 70.5, chromium sulphate at 82.5, and ferric sulphate at 89.1, while the value of SO_4 is held both by Kanonnikoff and myself to be 17.0. We may therefore deduce the following refraction-equivalents:—

Aluminium	9.7
Chromium	15.7
Iron	19.0

and :—

	From Aluminium Salt.	From Chromium Salt.	From Iron Salt.
Indium	17.7	17.2	17.2
Gallium.....	15.1	14.6	14.6

From the mean of the above the following constants may be derived :—

	Atomic weight.	Specific refraction.	Refraction-equivalent.
Indium	113.6	.1532	17.4
Gallium.....	69.8	.2120	14.8

These numbers can only be looked upon as approximate.

Turning to the matter of dispersion.—It has already been shown that the refraction-equivalent of an alum is the sum of the refraction-equivalents of its constituents for the line A. If this law holds good equally for the more-refrangible part of the spectrum, it follows that the dispersion-equivalent ($P \frac{\mu_G - 1}{d} - P \frac{\mu_A - 1}{d}$, or, which is the same thing, $P \frac{\mu_G - \mu_A}{d}$) of an alum is the sum of the dispersion-equivalents of its constituents. The data by which this can be tested are not so numerous or so trustworthy as in the former case, but the following may be accepted.

The dispersion-equivalent for water	0.212
” ” aluminium sulphate	2.40
” ” ammonium sulphate	1.33
” ” sodium sulphate	0.83

from which may be deduced :—

Dispersion-equivalent of	Calculated.	Observed.
Ammonium alum.....	8.62	8.40
Sodium alum	8.32	7.65

Though these figures are tolerably accordant, it will be seen that those in the first column are decidedly higher than those deduced from Soret's measurements. The differences are about 5 and 8 per cent. respectively; but there are known sources of error in experiment which may affect the first place of decimals.

The dispersion-equivalents of the different alums may be thus tabulated :—

	Dispersion-equivalents of the Alums.				
	Aluminium.	Chromium.	Iron.	Indium.	Gallium.
Ammonium salt ...	8.40	9.91	12.11		
Sodium ..	7.65				
Methylamine ..	8.47				
Potassium ..	7.92	9.73	11.31		
Rubidium ..	8.36	9.87	11.86	8.98	8.73
Cæsium ..	8.44	10.15	12.28		
Thallium ..	10.98	12.13	14.45		

It is evident at once :—

First. That the differences due to the replacement of one metal by another are very considerable; more considerable in proportion than the differences in the case of the refraction-equivalents.

Secondly. That the different compounds of the alkalis preserve the same order, and nearly the same proportion, in the aluminium, chromium, and iron series: the order is thallium far the highest, methylamine, cæsium, ammonium, rubidium, potassium, and sodium lowest. This is the same order which may be deduced from old observations on sulphates, nitrates, chlorides, and acetates.

Thirdly. That the order of the other metals is iron far the highest, chromium, indium, gallium, and aluminium lowest. This is also in accordance with observations on the simple sulphates of those previously examined.

Our knowledge on this part of the subject is not yet sufficiently advanced to determine the dispersion-equivalents of the separate elements.

PROCEEDINGS
OF
THE PHYSICAL SOCIETY
OF LONDON.

NOVEMBER 1885.

XXVI. *A Comparison of the Standard Resistance-coils of the British Association with Mercury Standards constructed by M. J. R. Benoit of Paris and Herr Strecker of Würzburg. By R. T. GLAZEBROOK, M.A., F.R.S., Fellow and Assistant Tutor of Trinity College, Cambridge*.*

IN accordance with the resolution of the Electrical Congress held in Paris in 1884, the Electrical-Standards Committee of the British Association decided to have made a series of resistance-coils in terms of the Legal Ohm; and the work of testing and comparing these coils was entrusted to me. In Paris, at the request of the Minister for the Post Office and Telegraphs, M. J. R. Benoit undertook the same task. An account of his experiments is published in the *Journal de Physique* for January 1885.

M. Benoit started *ab initio*, and constructed a series of glass tubes, the electrical resistance of which, when filled with mercury, can be calculated from their dimensions. Four of these were made, each having a resistance of about 1 ohm.

* Read May 23, 1885.

The plan adopted by the Committee was different. The specific resistance of mercury in terms of the British-Association unit has been carefully determined by Lord Rayleigh and others.

For the purpose of constructing Legal Ohms, it was decided to adopt a number for this quantity founded on their experiments, and then work from the British-Association standards. To adopt the other course would only have been to repeat at some trouble the experiments of Lord Rayleigh, M. Mascart, Herr Strecker, and others.

M. Benoit made a number of copies of his mercury-standards, and after reading his paper, I wrote and asked him to send me one or more of these, that I might compare them directly with the coils which had by that time been constructed for me by Messrs. Elliott Bros. M. Benoit replied most courteously, and sent me, about Easter, three of his copies. The object of the present paper is to give an account of the comparison of these with the standards belonging to the British Association.

M. Benoit's copies were mercury-standards. Each consisted of a glass tube bent several times, A A, fig. 1. The ends of this tube are ground flat and pass into two glass cups, B, B, which for most of their length are considerably wider than the tube, but taper down at their lower ends. The tubes are connected to the cups by short bits of india-rubber tubing, C, C, which fit tightly over the narrow ends of the cups, and are secured by wrapping string firmly over the india-rubber. To make the joint tight the india-rubber and glass were varnished over with shellac varnish. The cups open at the top, and can be closed by glass stoppers.

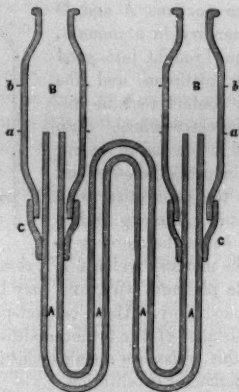
The standards were accompanied by full directions for use. I give a free translation of the greater part of the instructions.

1. On each of the cups are engraved two marks *a a*, *b b*. The lower mark *a a* indicates the position which the extremity of the tube introduced into the cup ought to occupy. The upper mark *b b* indicates the top of the mercury in the tube when filled. These two conditions must be fulfilled approximately; an error of .5 centim. in their sum changes the resistance either way by about .00018 ohm.

2. *Method of Filling.*—The tube may be filled in air in the

following manner:—Place it nearly horizontally and pour some mercury into one of the cups; then incline it gradually, shaking it a little, until the mercury enters the tube. The mercury should fill the tube slowly, without leaving a trace of air. Add mercury always by the same cup until it reaches the upper mark *bb* in both cups. The method is not always successful; it is better to work in a vacuum.

Fig. 1.



[Various methods for doing this are possible; it is perhaps hardly necessary to give the details of the one employed by M. Benoit.]

3. *Method of Using the Standard.*—The standard is introduced into the circuit by means of contact-pieces dipping into the mercury in the cups. These pieces should not reach more than a few millimetres below the surface. Contact-pieces of amalgamated copper cannot be used, because they render the mercury impure and alter its resistance rapidly, diminishing it appreciably in a few hours. Platinum, again, does not make a constant contact with the mercury; and the uncertainty produced by its variability is fatal to its use. To avoid this double difficulty, M. Benoit employed contact-pieces of a

special form (fig. 2). A glass tube A is drawn out to a point at its lower end; through this point a platinum wire B passes. The wire is held in position by shellac C; and a small cup D of thin glass is attached to its lower end also by shellac. Thus the two separate glass portions A and D are in electrical communication by means of the platinum wire B. The two portions A and D are filled with mercury in a vacuum. The mercury is thus brought into good contact with the platinum, and the mercury-platinum contact remains unchanged during the observations. Into the upper division A one end of a stout copper rod E is plunged; the other end of this rod forms the connection with the rest of the circuit. The cup D dips a few millimetres below the mercury in the standard. The mercury which fills it remains constantly in contact with the platinum wire, and may be kept without change almost indefinitely. These contact-pieces introduce an appreciable resistance of 2 to 3 thousandths of an ohm into the circuit; but this resistance remains constant, and can be eliminated by a method of substitution.

The standard is supported in a glass vessel, and can be reduced in temperature to zero by filling the vessel with finely-broken ice. The values of the standards are given at zero. The value at any other temperature is given by the formula

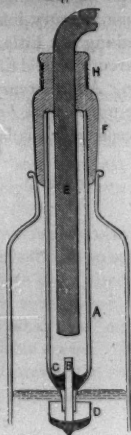
$$R_t = R_0(T + 0.0008649t + 0.0000112t^2),$$

t being the temperature in degrees Centigrade.

4. *Preparation of the Mercury.*—The mercury may be purified by the action of nitric acid, and dried by means of sulphuric acid and caustic potash.

Experiment showed that mercury from different sources, even when it had been rendered impure by the admixture of copper, lead, or zinc, after being treated by this process, gave

Fig. 2.



the same results. The process, however, does not free the mercury from silver or from the less oxidizable metals.

[In my experiments described below the mercury was freshly distilled by the aid of the admirable piece of apparatus designed by Weinhold, and introduced into England by Mr. W. N. Shaw.]

5. *Method of Cleaning the Tubes.*—The tubes should be cleaned by passing through them in succession:—

1. Distilled water.
2. Strong nitric acid.
3. Distilled water.
4. Ammonia.
5. Distilled water.

They must then be dried by a current of dry air.

To clean the tubes it is best to separate them from the cups. To effect this, the pieces of india-rubber tubing which connect them should be cut. To make the glass slip into the india-rubber tubing, it should be moistened with a drop of benzine.

In preparing the tubes for my measurements, I endeavoured to carry out the above instructions as carefully as possible. The resistance comparisons were made by Carey Foster's method, using the wire bridge of the British Association as designed by Dr. Fleming.

In order to compare the resistance of one of the tubes with a standard coil, it was necessary either to know accurately or to eliminate the resistance of the contact-pieces. It was also important to determine within what limits the resistance of the contact-pieces remained constant, and whether they could be repeatedly filled with mercury in such a way as to retain the same resistance.

For this purpose they were compared with a short piece of copper rod. The cups D (fig. 2), one belonging to each contact-piece, were immersed in the same beaker of mercury, being placed close together and covered to the same depth as when actually in use; the other ends of the copper rods E connected in the usual way to the bridge. The contact-tubes had been filled by being placed in a large test-tube which was exhausted, and into which mercury was then

admitted by means of a tap attached to the test-tube. The resistance of the contact-pieces was thus found in terms of the divisions of the bridge-wire before and after a set of comparisons of the mercury-tubes and standard coils.

It was found that the resistance of the contact pieces remained very nearly constant during the time occupied in making a series of comparisons. The variations in the values found before and after rarely amounted to more than 1 bridge-wire division, or about $\cdot 00005$ ohm; and the mean of the two values obtained is pretty certainly correct to less than this.

At the same time different fillings of the same contact-tubes led to very different values for their resistances. This was no doubt due to small irregularities in the contact between the platinum and the mercury. Thus I found the following values in terms of a bridge-wire division:—

Date &c.	Resistance of contacts.
March 25	98·4 b.w.d.
" Contacts refilled	69·4 "
" 26 { Copper rods being left in the } { mercury overnight..... }	68·2 "
" " Contacts refilled, one renewed	114·5 "
" 28 Copper rods being left in since 26th...	109·4 "

The differences between the 69·4 and the 68·2, and, again, between the 114·5 and 109·4, are due, I believe, to the copper of the contact-rods being dissolved by the mercury in the tubes.

On March 28th, the contact-tubes were cleaned and refilled, and the resistance was found to be 57·2 b.w.d. Throughout the experiments until April 9, when one of them was broken, their values did not differ greatly from the above.

On April 11th, new contact-tubes were prepared and used throughout the rest of the experiments; their value remained fairly constant at about 58 b.w.d.

Thus the quantity actually determined by the observations was the difference between the resistance of the mercury tube + the contact-pieces and the standard coil. In order to get

the resistance of the mercury tube in terms of the standard coil, the resistance contact-pieces found as above described had to be subtracted.

Another way of eliminating the resistance of the contacts from the result is to compare first the mercury tube and then the standard with some other coil, using the same contact-pieces to connect the mercury tube and the standard to the bridge. This can be done by removing the contact-tubes from the mercury tube, and dipping them into two mercury cups, into which the electrodes of the standard also dip. Since, however, my apparatus allowed me to measure the resistance of the contact-tubes with all the accuracy I desired, the first method was the more convenient, as it necessitated fewer observations.

The comparisons were made both with the original standards of the British Association—B.A. Units—and also with the Legal-Ohm Standards of the same, recently constructed for me by Messrs. Elliott Bros., taking as a basis the value for the resistance of mercury in terms of the B.A. unit adopted by the British-Association Committee. According to this,

$$1 \text{ Legal Ohm} = 1.0112 \text{ B.A.U.}$$

The three mercury tubes Nos. 37, 38, 39, sent by M. Benoit, were compared. Between two observations of resistance the mercury was occasionally drawn through the tubes by connecting the cups alternately to the air-pump. I hoped in this manner to displace any very small air-bubble or particle of dust which might have got lodged in the tube. In the final set of observations recorded below, it will be seen that no appreciable change in the resistance was detected by this precaution. I therefore suppose the tubes to have been properly filled.

In some earlier observations, which will be referred to again shortly, variations of considerable amount were produced by passing the mercury through. In the tables below the values of the resistances which were found by comparison with the B.A.U., and reduced to Legal Ohms by means of the known relation between the two, are denoted by an asterisk.

TABLE I.

Tube No. 37. Value given by Benoit, 1.00045.	
Date and Method of Treatment.	Value in L. Ohms.
April 13, 10 A.M.	.99991*
" 10.30	.99987
" 11.55	.99998
" 1 P.M. { After passing the } mercury through. }	.99997
April 14	.99998
April 14 { After passing the } mercury through. }	.99996
April 15	.99979
April 15	.99976*
Mean value.....	.99990

TABLE II.

Tube No. 38. Value given by Benoit, 1.00066.	
Date and Method of Treatment.	Value in Legal Ohms.
April 8, 11.45 A.M.	1.00006
April 8, 1.30 A.M.	1.00018
April 9	1.00023*
April 9	1.00019
April 11 { Tube taken to pieces, re- } cleaned and refilled. }	
April 13	1.00011*
April 13	1.00010
April 14	1.00015
April 15	1.00002
April 15	1.00003*
Mean value.....	1.00012

TABLE III.

Tube No. 39. Value given by Benoit, .99954.	
Date and Method of Treatment.	Value in Legal Ohms.
April 14	.99905
April 15, 10.30 A.M.	.99917
April 15, 11 A.M. { After passing the } mercury through. }	.99921
April 15, 11.55 A.M.	.99921
April 15, 12.15 { After passing the } mercury through. }	.99921
April 15	.99917
Mean value.....	.99917

Thus we may collect the results as below.

TABLE IV.

No. of Tube.	Value found by Benoit.	Value found by R. T. G.	Difference.
37.....	1·00045	·99990	·00055
38.....	1·00066	1·00011	·00055
39.....	·99954	·99917	·00037
Mean.....	1·00022	·99972	·00050

Thus there is a difference of ·0005 legal ohm between the two. My results are based upon the value of the resistance of mercury in terms of the B.A. unit adopted by the British-Association Committee. If we denote by M the resistance at 0° C. of a column of mercury 1 metre long, 1 square millim. in section, then, according to the number adopted by the Committee,

$$M. = \cdot 9540 \text{ B.A.U.}$$

$$\text{B.A.U.} = 1\cdot 04820 \text{ M.}$$

The most recent values actually found for these relations are given below:—

TABLE V.

M in B.A.U.	B.A.U. in M.	Observer.
·95412	1·04803	{ Rayleigh * and Sidgwick.
·95374	1·04851	
·95334	1·04894	{ Mascart†, Nerville, and Benoit.
·95411	1·04809	
		Strocker ‡.
		L. Lorentz §.

M . Benoit has not compared his tubes directly with the B. A. units. The mean of my comparisons gives for his mercury tubes,

$$M. = \cdot 95248 \text{ B.A.U.}$$

$$\text{B.A.U.} = 1\cdot 04878 \text{ M.}$$

* "On the Specific Resistance of Mercury," Phil. Trans. pt. 1, 1883.

† *Journal de Physique*, June 1884.

‡ *Abh. der bayer. Akad.* Bd. xv.; also *Wied. Ann.* vol. xxv.

§ *Wied. Ann.* vol. xxv.

Thus there is a difference of $\cdot 0005$ ohm between standards issued by M. Benoit as representing the legal ohm, and those issued by myself on behalf of the B. A. Committee, M. Benoit's standards being less than mine by that amount.

Some other points remain to be noticed. The first time that I filled Nos. 37 and 39, I obtained results for the resistances differing appreciably from those given above. They were filled each time with equal care, the mercury had been treated in the same manner, and there seemed no reason, taking each observation separately, why one should be better than the other. For No. 37 I found on April 8th the value $1\cdot 00044$ legal ohms; I noticed on April 9 that this had gone up to $1\cdot 00071$; and on passing the mercury through as already described, it rose further to $1\cdot 00080$. These differences are considerably greater than any possible errors of observation; and it was clear that there was some small change progressing in the tube. It was carefully examined, but no trace of an air-bubble could be seen; and then it was emptied, cleaned again, and refilled, with the results given in Table I. Much the same was observed with coil No. 39. No. 38 was cleaned and filled twice, but gave perfectly consistent results all through. This uncertainty seems to me to constitute one serious objection to the general employment of mercury tubes as standards.

I believe that for the observations recorded in the above Tables the tubes were properly filled, and that there were no bubbles of air in them. This, I think, is shown by the agreement between the results of different fillings, and I suppose that the resistance will not seriously alter so long as the same mercury remains in the tubes; but it appears that, after filling, electrical experiments are required to make certain that every bubble of air has been removed, and that the tube really has its true resistance. I should imagine, too, that the tubes would require somewhat frequent refilling to make sure that the mercury may remain pure. My experiments showed that it was almost impossible to keep the inside of the cups above the mercury perfectly dry. It was necessary, when making the comparisons, to remove the glass stopper and insert the contact-pieces; and this had to be done when the tube was immersed in the ice-water.

The cups were open to the air for a short time in making this change, and that short time was quite sufficient to cause a deposit of dew to be formed on the inside of the cup. So long as this slight moisture remained in the cup and did not reach the tube itself, of course it did not affect the results; but it would be difficult to feel certain that after a time the mercury in the tube was quite dry.

This difficulty would be avoided by working at the temperature of the room rather than at that of melting ice. The large temperature-coefficient of mercury, from three to four times that of platinum-silver alloy, is, however, an objection to this.

Another difficulty, caused by the necessity of working at zero, was that the mercury in the cups was always slightly warmer, $0^{\circ}.2$ or $0^{\circ}.3$ C., than the ice. This was, no doubt, caused by the conduction of heat down the copper connecting-rods, and from the upper portions of the glass of the cups which were exposed to the air. This, of course, would necessitate a small correction to the values of the resistances given in the Tables; but the correction must be exceedingly small, for the temperature of the mercury in the tube itself, where it is actually in contact with the ice, must be zero, and it is only the small portions of the tubes which lie within the cups which will be at the higher temperature.

The necessity of using the somewhat complicated connecting pieces is also a great drawback to the practical usefulness of the mercury standards. I hope shortly to carry out some experiments on the permanence of the contact between mercury and amalgamated platinum. Contact-pieces of platinum with their end amalgamated with mercury would be easier to work with, and should give consistent results.

I have also made some observations on the variation of the resistance of mercury with temperature.

The formula quoted above from Benoit is given by Mascart, Nerville, and Benoit as determined from experiments between 0° and 100° .

It is

$$R_t = R_0(1 + .0008649t + .00000112t^2).$$

According to Strecker,

$$R_t = R_0(1 + \cdot 000900t + \cdot 00000045t^2).$$

This is derived from observations at 0° , 10° , 15° , and 20° .

Lorenz finds that between 0° and $27^\circ\cdot 32$ his experimental results agree with the formula

$$R_t = R_0(1 + \cdot 0009013t);$$

and from $8^\circ\cdot 32$ to $35^\circ\cdot 31$ with

$$R_t = R_0(1 + \cdot 000916t).$$

While Siemens and Halske give

$$R_t = R_0(1 + \cdot 0008523t + \cdot 000001356t^2).$$

I made observations on the two tubes, Nos. 37 and 39, determining their resistances at temperatures of about 0° , 5° , 10° , and 15° ; and the results of the observations show that the average change for low temperatures is less than that given by the above formulæ. Thus, if we call $(R_t - R_0)/R_0t$ the average change between temperatures t and 0 , the values, as found from my experiments and as calculated by the formulæ of Siemens and Benoit, are as given below.

Average change between	Siemens and Halske.	Benoit.	R. T. G.
5° and 0°	$\cdot 000858$	$\cdot 000870$	$\cdot 000834$
10° „ 0°	$\cdot 000865$	$\cdot 000876$	$\cdot 000861$
15° „ 0°	$\cdot 000872$	$\cdot 000881$	$\cdot 000879$

The values given in the last column are the mean of those derived from the two tubes; the greatest difference between the mean and any one observed value was $\cdot 000005$. Strecker's values would be above Benoit's.

Lord Rayleigh found for the average change between 0° and 12° , $\cdot 000861$.

Of course the value of this temperature-coefficient depends somewhat on the glass of the tube; but the differences between the coefficients of expansion of various kinds of glass are too small to account for the whole of the differences shown above.

The paper by Herr Strecker on the same subject has been already referred to, and it will be noticed that the number he arrives at as expressing M. in B.A. units differs by $\cdot 00078$ from that given by Lord Rayleigh; or, if we take the numbers expressing the B.A.U. in terms of M., the difference is $\cdot 00086$. Now Strecker had compared his mercury tubes with a B.A.U. tested carefully by myself against the standards, and found to have the value of

$$\cdot 99937 \text{ B.A.U. at } 17\cdot 7.$$

This coil is marked L , No. 54.

On the conclusion of his experiments, Herr Strecker returned this coil to me, and kindly sent with it a German-silver copy of his tubes marked No. 13, and said to have, according to his determinations, a resistance of

$$1\cdot 00189 \text{ M. at } 10^{\circ} \text{ C.,}$$

with a temperature-coefficient of

$$\cdot 000247 \text{ per degree.}$$

It was thus in my power to repeat my comparison of L , No. 54, with the standards, and to compare the values of M. as found by Herr Strecker with the B.A.U. The value found for L 54 agreed very closely with that given by Herr Strecker; the results of my comparison of Herr Strecker's coil with the B.A.U. are given below. The comparisons were made as usual, by Carey-Foster's method; but since the difference between the two resistances was such as to require the use of a long piece of the bridge-wire, a resistance of 20 B.A. units was introduced in multiple arc with the standard 1 unit. This resistance was composed of the two standard ten-unit coils of the Association.

TABLE VI.

Date.	Standard of Comparison.	B.A.U. in M.	M. in B.A.U.
April 9	Flat.	1.04814	.95407
" 15	♁ 54.	1.04806	.95413
" "	Flat.	1.04791	.95427
" 16	Flat.	1.04820	.95402
" "	♁ 54.	1.04811	.95410
" 17	♁ 54 and 20.	1.04801	.95419
" "	Flat and 20.	1.04824	.95399
" "	Flat and 20.	1.04797	.95422
Mean		1.04808	.95412

These numbers, it will be seen, agree with those found by Lord Rayleigh and Lorenz, and differ greatly from Strecker's values.

They represent the results of the comparison of a German-silver copy of Strecker's tubes with the B.A. units, including the actual coil employed by Strecker, ♁ No. 54; while Strecker's numbers give the result of the comparison of his tubes directly with No. 54.

I have not been able to explain the discrepancy. Herr Strecker suggests in a letter to me on the subject, that the resistance of his coil No. 13 has changed somewhat in the year which has elapsed since his comparisons were made. At present I am in correspondence with Prof. F. Kohlrausch, of Würzburg, in whose laboratory Strecker's experiments were conducted, and hope to obtain another copy of Strecker's tubes, or at any rate to have another comparison made between his tubes and the B.A. standards.



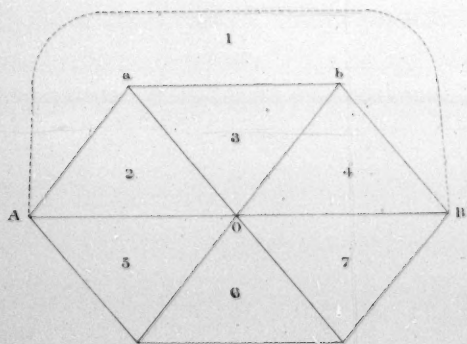
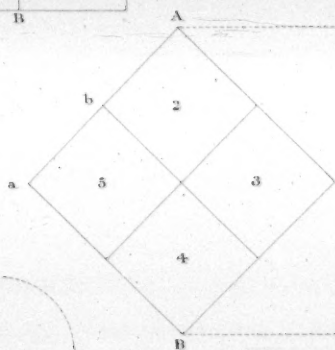
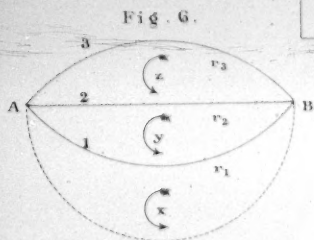
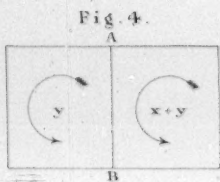
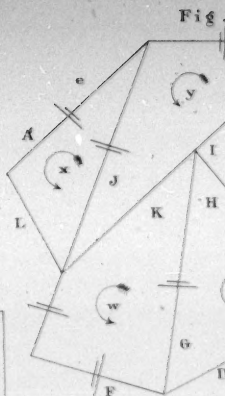
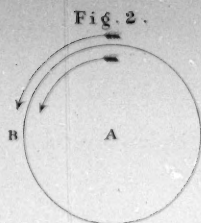
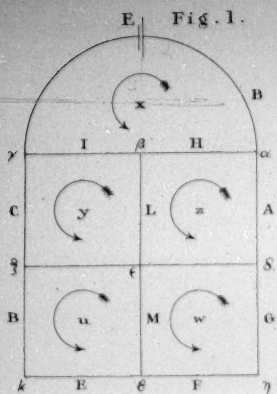


Fig. 8.

Fig. 3.

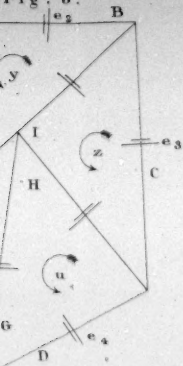


Fig. 5.

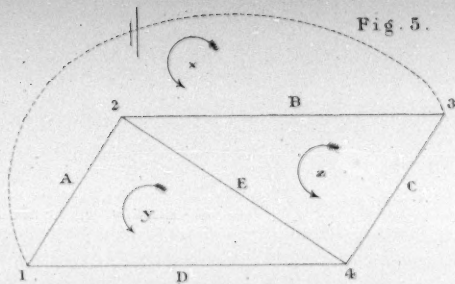


Fig. 9.

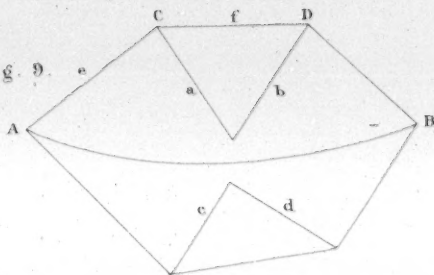


Fig. 7.

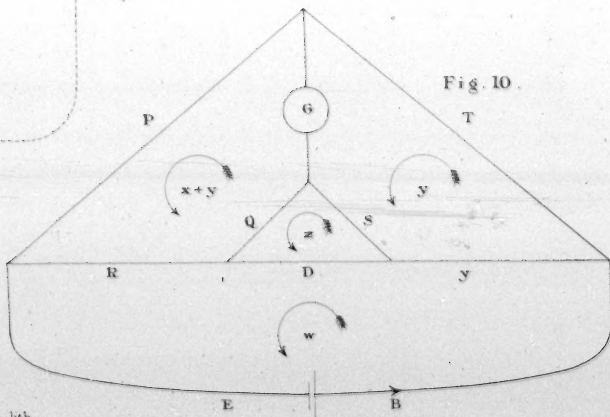


Fig. 10.

Fig. 11.

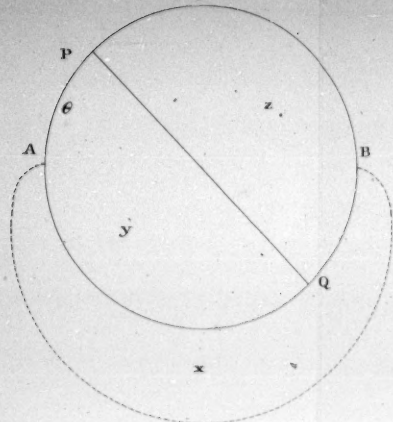


Fig. 16.



Fig. 17.

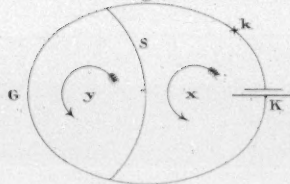


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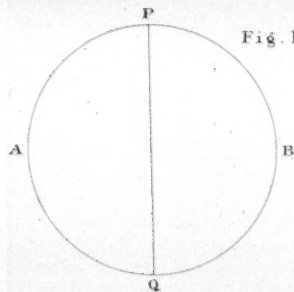


Fig 12

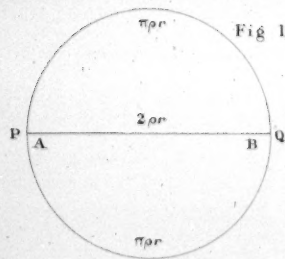


Fig. 14.

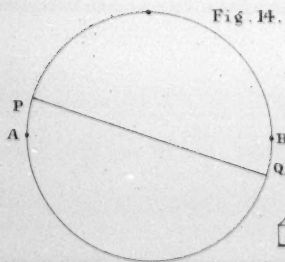


Fig. 15.

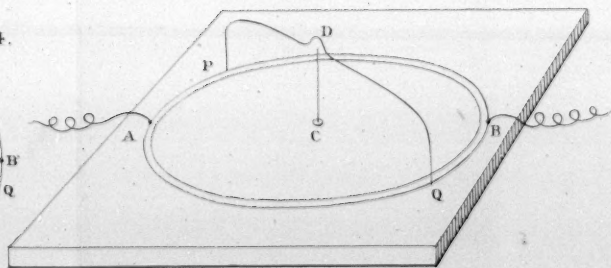


Fig. 18.

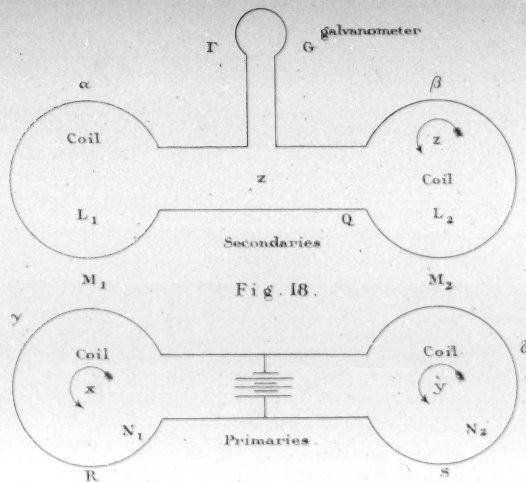
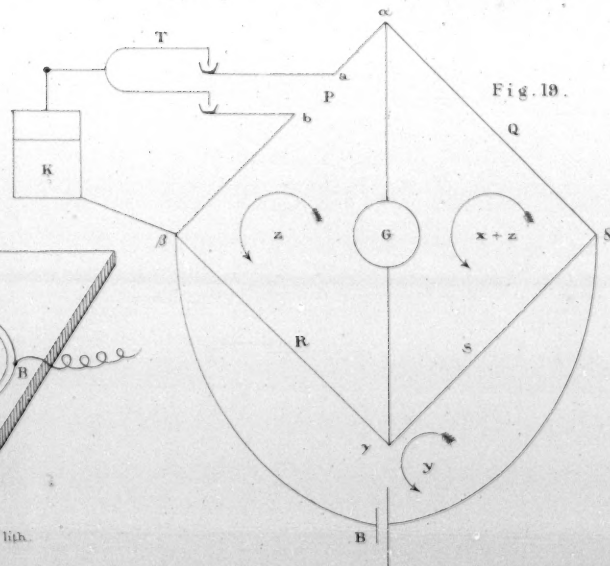


Fig. 19.



XXVII. *Problems on the Distribution of Electric Currents in Networks of Conductors treated by the Method of Maxwell.* By J. A. FLEMING, M.A., D.Sc. (Lond.), Fellow of St. John's College, Cambridge, Professor of Electrical Technology in University College, London*.

[Plates III. & IV.]

If any number of points in a plane be joined together by linear conductors such as metallic wires, we have an arrangement of conductors which is called a Network. If at any point in the network a current of electricity be allowed to flow in and is drained off at some other point by conductors, called respectively the anode and kathode conductor, then, after a short period, depending on the self and mutual induction coefficients of the various conductors, the total quantity of electricity arriving by the anode will distribute itself throughout the network and settle down into a steady flow. When this is the case there is a certain definite difference of potential between the anode or source-point and the kathode or sink-point, and there is also a certain definite and constant strength of current in the anode conductor and in every mesh or branch of the network. Call α and γ the potentials of these source- and sink-points, and x the strength of the current in the anode lead, that is the whole quantity of electricity flowing per second through the network, then $(\gamma - \alpha)/x$ measures the resistance of the network. We can imagine the network replaced by a single linear conductor or wire of such sort that if the anode and kathode conductors are applied to its ends, the difference of potentials at the ends of this simple conductor and the strength of the current flowing through it have the same numerical values γ , α , and x . The resistance of this single conductor is then the same as that of the complex network.

The resistance of the network is obviously some function of the resistances of the separate conductors or wires which compose it, and is capable of being calculated from them. Experimentally, the resistance of a complicated network would

* Read June 27, 1885.

best be determined by the measurement of the current-strength in the anode lead and the difference of potential between the source and the sink. Theoretically, it is interesting to examine the law of distribution of currents in a network, and to reduce to a function of the separate resistances the total resistance of the whole network between any two points.

§ 2. In his larger Treatise on Electricity, Clerk Maxwell has treated the general case to determine the differences of potentials and the currents in a linear system of n points connected together in pairs by $\frac{1}{2}n(n-1)$ linear conductors*, and has shown how to form the linear equations, the solution of which gives the condition of the network when given electromotive forces acting along some or all of the branches have established steady currents in them.

The usual method of obtaining a solution for the distribution of currents is the application of Ohm's law round the several currents of the network, controlled by the condition of continuity that there is no creation nor destruction of electricity at the junctions.

Since the publication of the first edition of his Treatise, Maxwell reduced these two sets of equations to one set by the simple device of regarding the real currents in the meshes of the network as the *differences* of imaginary currents round each cycle or mesh of the network, all directed in the same direction, and thus obtained by the application of Ohm's law a single set of linear equations, the solution of which gives the required currents in each branch. Maxwell's method is as follows†:—If we have p points in space and join them together by lines, the least number of lines which will connect all the points together is $p-1$. If we add one line more we make a closed circuit somewhere in the system; that is

* 'A Treatise on Electricity and Magnetism,' 2nd edition, Vol. i. § 280 and § 347.

† This method was first given by Clerk Maxwell in his last course of University lectures. It is alluded to in the second edition of his larger Treatise and in the Appendix of his smaller Treatise by their respective editors, Mr. W. D. Niven and Professor Garnett, to whom it was communicated by the present writer.

to say, a portion of space is enclosed and forms a cell cycle or mesh. Every fresh line added then makes a fresh mesh; and hence if there are l lines altogether joining p points, the number of cycles or cells will be $k = l - (p - 1)$. Now let such a system of points and lines represent conducting wires joining fixed points, and forming a conducting network. Let a symbol be affixed to each point which represents the electrical potential at that point, and also a symbol affixed to each line representing the electrical resistance of the conductor represented by it. In such a diagram of conductors the form is a matter of indifference so long as the connections are not disturbed and lines are not made to cross unless the conductors they represent are in contact at that point.

Consider a network, Pl. III. fig. 1, formed by joining nine points by thirteen conductors. Then there will be $13 - (9 - 1) = 5$ cycles or cells. Now let an electromotive force E act in one branch B , and give rise to a distribution of currents in the network. Let $\alpha, \beta, \gamma, \delta$, &c. represent the potentials at the points, and A, B, C, D , &c. the electrical resistances of the conductors joining these points, and imagine that round each cycle or circuit an imaginary current flows, all such currents flowing in the same direction.

A circuit is considered to be circumnavigated positively when you walk or go round it so as to keep the boundary on your right hand. Hence, going round an area A in the direction of the arrow is positive as regards the inside if you walk inside the boundary-line, and negative as regards external space B if you walk in the same direction round the outside. We shall consider a current, then, as positive when it flows round a cycle in the opposite direction to the hands of a watch. Returning then to the network, we consider that round each cycle flows an imaginary current in the positive direction. The real currents in the conductors are the differences of these in adjacent cycles or meshes, and the imaginary currents will necessarily fulfil the condition of continuity, because any point is merely a place *through* which imaginary currents flow, and at which therefore there can be no accumulation nor disappearance of electricity.

Let x, y, z , &c. denote these imaginary like-directed currents. Then $x - y$ denotes the real current in the branch I ,

and similarly $x-z$ that in branch H. Then x, y, z , &c. may be called the cyclic symbols of these areas. The cyclic symbol of external space is taken as zero; hence the real current in branch B is simply x .

Let an electromotor act on the branch B, bringing into existence an electromotive force in that branch. Let the internal resistance of the electromotor be included in the quantity B, representing the resistance of the branch A. Then apply Ohm's law to the cycle x formed by the conductors B, I, H; we have

$$E - Bx = \gamma - \alpha.$$

x is the actual current in this case flowing in the resistance B, and the potential at the ends of B is equal to the effective electromotive force acting in it *less* the product of the resistance of the conductor multiplied by the current flowing in it. For the conductor I we have similarly

$$\gamma - \beta = (x - y)I.$$

Hence $x - y$ represents the actual current in I: it is the difference of the imaginary currents flowing round the x and y cycles in the positive direction. And for the conductor H we have also

$$\beta - \alpha = (x - z)H.$$

Add together these three equations,

$$E = \gamma - \alpha + Bx,$$

$$0 = \beta - \gamma + (x - y)I,$$

$$0 = \alpha - \beta + (x - z)H;$$

and we have, as the result of going round the cycle x formed of conductors B, I, and H,

$$E = x(B + I + H) - yI - zH. \quad \dots \quad (1)$$

α, β, γ have disappeared in virtue of these opposite signs.

This equation (1) is called the equation of the x cycle; and we see that it is formed by writing as coefficient of the cyclic symbol x the sum of all the resistances which bound that cycle, and subtracting the cyclic symbol of each neighbouring cycle multiplied respectively by the common bounding resistance as coefficient, and equating this result to the effective electromotive force acting in the cycle, written as positive or nega-

tive according as it acts with or against the imaginary current in the cycle. This is Maxwell's rule.

Since there are k cycles or meshes we can in this way form k independent equations, and by the solution of these determine the k independent variables, x, y, z , &c. The value of the current in any branch is then obtained by simply taking the difference of these variables belonging to the adjacent meshes, of which the conductor or branch considered is the common boundary.

§ 3. Let us now consider the most general case possible, in which we have a network composed of linear conductors sufficiently far apart to have no sensible mutual induction, and let there be electromotive forces acting in each branch or conductor. Let the system be considered to have arrived at the steady condition. Let x, y, z , &c. be the cyclic symbols or measure of the imaginary current circulating counterclockwise round each mesh. Let A, B, C , &c. (fig. 3) be the resistances, and e_1, e_2, e_3 , &c. the electromotive forces acting in each branch. These are reckoned positive when they tend to force a current round the mesh counterclockwise, and negative when they act in the opposite direction. Then the equation to the x cycle will be

$$x(A + J + L) - yJ + Oz + Ou + Ow = e_1.$$

The symbols of all the cycles are written down, putting in those of z, u , and w with zero coefficients, as they are not adjacent cycles to that of x . We shall have five equations similar to the above for the other cycles, y, z, w , and u .

Now it can very simply be shown from the theory of determinants, that if there are n linear equations of the type

$$\begin{array}{ccccccc} a_1x_1 + a_2x_2 + & . & . & . & . & . & a_nx_n = p_1, \\ b_1x_1 + b_2x_2 + & . & . & . & . & . & b_nx_n = p_2, \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ k_1x_1 + k_2x_2 + & . & . & . & . & . & k_nx_n = p_n, \end{array}$$

the solution for any variable x_1 is the quotient of the determinants

$$x_1 = \frac{\begin{vmatrix} p_1 & a_2 & \cdot & \cdot & \cdot & a_n \\ p_2 & b_2 & \cdot & \cdot & \cdot & b_n \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ p_n & k_2 & & & & k_n \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 & \cdot & \cdot & \cdot & a_n \\ b_1 & b_2 & \cdot & \cdot & \cdot & b_n \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ k_1 & k_2 & & & & k_n \end{vmatrix}}$$

The only difference between the numerator and denominator is that the solution for x_n is given by writing as numerator the determinant of the n equations having the column $p_1, p_2 \dots p_n$ substituted for its n th column, and then writing down as denominator the determinant of the n equations simply.

Thus, for example, the solution of the three linear equations

$$ax + by + cz = d,$$

$$a_1x + b_1y + c_1z = d_1,$$

$$a_2x + b_2y + c_2z = d_2,$$

is

$$x = \frac{\begin{vmatrix} d & b & c \\ d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}$$

with similar expressions for y and z , differing only in having as numerators respectively

$$\begin{vmatrix} a & d & c \\ a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} a & b & d \\ a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \end{vmatrix},$$

denominator being the same.

In this case the evaluation of these determinants is easy : a

simple symmetrical process of taking products, according to the rule,

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = (aei + bfg + cdh) - (ecg + bdi + afh).$$

§ 4. The properties of determinants enable us, however, very easily to evaluate a numerical determinant of any order. The process consists in the gradual reduction of the determinant in order by such transformations as will render all the elements of the first row or column zero except the first. The determinant is then reduced to the product of its leading elements and the corresponding minor. A repetition of this lowers the determinant one degree at each stage; and finally, when it is resolved into a numerical two-row determinant, a simple cross multiplication gives its value.

The process of evaluation of a numerical determinant is dependent on four principles:—

(1) That the value of a determinant is not altered if rows are changed into columns.

(2) The interchange of two rows or two columns reverses the sign of the determinant.

(3) If every constituent in any row or column be multiplied by the same factor, then the determinant is multiplied by that factor.

(4) A determinant is not altered if we add to each constituent of any row or column the corresponding constituents of any other row or column multiplied respectively by an identical factor, positive or negative.

For example, suppose that the solution of a series of network equations with numerical coefficients of resistance yield the determinant

$$\begin{vmatrix} 5 & 3 & 1 & 6 \\ 7 & 8 & 9 & 2 \\ 2 & 1 & 4 & 3 \\ 10 & 7 & 5 & 7 \end{vmatrix},$$

we proceed to operate on this as follows:—Subtract the second column from the first and write the remainder. As a new first column we get

$$\begin{vmatrix} 2 & 3 & 1 & 6 \\ -1 & 8 & 9 & 2 \\ 1 & 1 & 4 & 3 \\ 3 & 7 & 5 & 7 \end{vmatrix}.$$

Subtract the third row from the first and put the remainder as a new first row. Also add the third row to the second for a new second row, and we get

$$\begin{vmatrix} 1 & 2 & -3 & 3 \\ 0 & 9 & 13 & 5 \\ 1 & 1 & 4 & 3 \\ 3 & 7 & 5 & 7 \end{vmatrix}.$$

Again, subtract the first row from the third for a new third, and subtract three times the first row from the fourth row for a new fourth row, and we have

$$\begin{vmatrix} 1 & 2 & -3 & 3 \\ 0 & 9 & 13 & 5 \\ 0 & -1 & 7 & 0 \\ 0 & 1 & 14 & -2 \end{vmatrix}$$

which is equivalent to the third order determinant

$$\begin{vmatrix} 9 & 13 & 5 \\ -1 & 7 & 0 \\ 1 & 14 & -2 \end{vmatrix}.$$

And a similar series of operations reduces this to

$$\begin{vmatrix} 76 & 5 \\ 21 & -2 \end{vmatrix}$$

which is equal to

$$-76 \times 2 - 5 \times 21 = -257.$$

Accordingly a series of simple subtractions and multiplications will effect the evaluation of any numerical determinant, and enable us to solve a series of linear network equations for the currents in all the branches when the numerical values of the resistances of the conductors are given. The equations as written above give as solutions the values of the cyclic symbols or imaginary currents round each mesh. To obtain the actual current in any branch, we should have to obtain the values of the cyclic symbols or imaginary currents, for the adjacent meshes of which the given branch is a common

boundary. Maxwell ingeniously saves labour in this operation by taking as the symbol for one mesh say $x+y$, and for an adjacent mesh y (fig. 4), and then the real current in the branch AB is

$$x+y-y=x.$$

And the simple rearrangement and solution of the network equation gives at once as value for x the current in the resistance AB, which is the common partition of the two meshes.

§ 5. Returning now to the case when there is only one impressed electromotive force in one branch, we see that in forming the cycle equations only one will be equated to an electromotive force, viz. the equation for the mesh containing the impressed electromotive force in one of its branches. All the other equations will be equated to zero; and accordingly the equation for the current in any conductor will be of the form

$$x = \frac{E\Delta_{n-1}}{\Delta_n};$$

where Δ_n is a determinant of the n th order, and Δ_{n-1} is a first minor of this. Referring to fig. 1, we see that, by writing down the five equations of the cycles x, y, z, u, w , we obtain equations by which to calculate the currents in any of the thirteen branches, and the current in branch B will be

$$x = \frac{E\Delta_{n-1}}{\Delta_n};$$

where Δ_n is the determinant formed of the coefficients of the five equations, and Δ_{n-1} is the first minor corresponding to the coefficient of x in the equation of the x -cycle.

We also saw that if γ and α are the potentials at the ends of the branch B, $E - Bx = \gamma - \alpha$.

Now consider that part of the network which remains if the conductor B is removed, and let us imagine that a current n continues to be forced into it at γ and drained out at α ; the total resistance of that part of the network, not counting B, is

$$\frac{\gamma - \alpha}{x};$$

but this is equal to

$$\frac{E}{x} - B.$$

Now since the resistance of B may be anything, let it be zero;

then the total resistance of the network between γ and α will be

$$R = \frac{E}{x};$$

but

$$x = \left[\frac{E \Delta_{n-1}}{\Delta_n} \right]_{B=0},$$

where the suffix and bracket denote that after the determinants are formed from the cycle equations, according to Maxwell's rule, then in them B is put equal to zero.

If we denote the determinant of all the n -cycle equations under the condition of $B=0$ by d_n , and by d_{n-1} the first minor of this or the minor of its leading element corresponding to the coefficient of x with the resistance of the circuit containing the effective electromotive force put equal to zero, we have for the total resistance R of the network between the points at which the current enters and leaves, the expression

$$R = \frac{d_n}{d_{n-1}}.$$

Since, then, as we have seen, the linear equations for the cycles can always be solved by evaluating the determinants, it follows that in all cases, no matter how complicated, the resistance of any network can be calculated by simple arithmetic processes from the given resistances of the branches or conductors which compose it. We have therefore an interesting extension of Maxwell's method of calculating the currents in a network and the potentials at the junctions to a method of calculating the combined resistance of a number of conductors forming a network; which method consists, as seen above, in forming a certain determinant whose elements are formed of the separate resistances of the branches, and dividing this determinant by another of an order next below, viz. the first minor of its leading elements; and we find that the resistance between any two points of any network of conductors, however complicated, is expressible as the quotient of a certain determinant by another formed from it.

§ 6. We shall proceed to illustrate this method by a few examples.

1. Find the resistance between the points 1 and 3 (fig. 5) of a network consisting of five conductors, whose resistances are A, B, C, D, E , joining four points, 1, 2, 3, and 4.

Connect 1 and 3 by an imaginary conductor of zero resistance, and having an electromotive force, e , supposed to act in it. Let x, y, z denote the cycles or imaginary like-directed currents in the three meshes so formed, and write down the current equations, according to Maxwell, for these three cycles:—

$$\begin{aligned}(A+B)x - Ay - Bz &= e, \\ -Ax + (A+E+D)y - Ez &= 0, \\ -Bx - Ey + (B+C+E)z &= 0.\end{aligned}$$

Then, by what has been shown above, the resistance R between the points 1 and 3 of the network is given by the expression

$$R = \frac{\begin{vmatrix} (A+B), & -A, & -B \\ -A, & (A+E+D), & -E \\ -B, & -E, & (B+C+E) \end{vmatrix}}{\begin{vmatrix} (A+E+D), & -E \\ -E, & (B+C+E) \end{vmatrix}}$$

In dealing with numerical cases we need no longer introduce any notice of imaginary electromotive forces, but proceed according to the following rule.

To determine the resistance of a network of conductors between any two points on the network. Join these two points by a line whose resistance is supposed zero, and give symbols to the meshes of the network so formed; calling this additional mesh produced by the added zero conductor the *added mesh*. Then write down a determinant whose dexter diagonal has for elements the sum of the resistances which bound each mesh, beginning with the added mesh; and for the other elements of each row the resistances which separate this mesh respectively from adjacent meshes, and having the minus sign prefixed, zeros being placed for elements corresponding to nonadjacent meshes.

More explicitly, if we denote by x, y, z , &c. the meshes, x being the added mesh, and by $\Sigma R_x, \Sigma R_y, \Sigma R_z$, &c. the sum of the resistances which bound each cycle, then these will be the elements along the dexter diagonal of the determinant.

And if x and y are adjacent meshes, and R represents the resistance of the common boundary, then $-R$ will be the element in the x th row and y th column, and also in the y th row and x th column; but if x and z are nonadjacent meshes, then 0 will be the element in the x th row and z th column, and also in the z th row and x th column. Having formed this determinant, which we call the network determinant, we divide it by the first minor of its leading element; and the quotient is the resistance of the network between the two points, joined by the zero-conductor forming the added mesh. It is seen that, owing to the mode of formation of the network equations, the network determinant is a symmetrical determinant—that is, one half of the determinant is the reflection, as it were, of the other half in the diagonal considered as a mirror.

§ 7. As a means of comparing the results of this method with other known results, let us take the exceedingly simple case of three conductors joining two points in what is commonly called multiple arc.

Let 1, 2, and 3 (fig. 6) be the three conductors joining two points A and B; let their respective resistances be r_1, r_2, r_3 ; then join A, B by a dotted line so as to make one added mesh, and let the resistance of this added circuit be zero. Then, without writing down the equations to the cycles, we see that the network determinant is

$$d_n = \begin{vmatrix} r_1 & -r_1 & 0 \\ -r_1 & r_1 + r_2 & -r_2 \\ 0 & -r_2 & r_2 + r_3 \end{vmatrix}$$

The elements $r_1, r_1 + r_2, r_2 + r_3$ of the dexter diagonal are the sums of the resistances which bound each mesh, x, y , and z , taking the added mesh x first.

The other elements of the first row are the resistances, with minus sign prefixed, which separate the mesh x from mesh y and mesh z ; or are common to x and y and x and z , viz. r_1 and zero, because x and z are nonadjacent. And, similarly, if m and n are any two meshes, then the element in the n th row and m th column is the resistance separating or common to the two meshes; and the element in the n th row and m th column is identical with that in the m th row and n th column:

zero being placed as an element if these meshes, m and n , have no common boundary or circuit.

The above determinant is easily evaluated. By adding the first row to the second for a new second row, and this new second row to the third for a new third row, we transform the determinant easily into

$$\begin{vmatrix} r_1 & -r & 0 \\ 0 & r_2 & -r_2 \\ 0 & 0 & r_3 \end{vmatrix}$$

which is equal to

$$r_1 r_2 r_3.$$

The first minor of the leading term of the network determinant is

$$\begin{vmatrix} r_1 + r_2 & -r_2 \\ -r_2 & r_2 + r_3 \end{vmatrix} = d_{n-1},$$

which is equal to

$$r_1 r_2 + r_2 r_3 + r_3 r_1;$$

and hence the resistance of the network between A and B is

$$\frac{d_n}{d_{n-1}} = \frac{r_1 r_2 r_3}{r_1 r_2 + r_2 r_3 + r_3 r_1},$$

which is a known result. In these simple cases the above general rule is, of course, a less easy method of finding the combined resistance than the direct application of Kirchhoff's corollaries of Ohm's law; but whereas the general method is alike applicable to the most complicated as well as to the most simple cases, the simple direct method requires twice as many equations, and does not determine the direction as well as magnitude of the current in each branch.

§ 8. As a simple numerical example we may take the case of a crossed square of wires. Let 12 conductors join 9 points (fig. 7) so as to form a square divided into four squares, or a four-mesh network of conductors. Let the resistance of each branch, as ab , be unity. It is required to find the combined resistance between A and B. Number the meshes 1, 2, 3, 4, 5; 1 being the added mesh formed by joining AB by a dotted line, making an additional fifth mesh, the resistance of this additional ideal conductor being zero. Then the network

determinant is

$$\begin{vmatrix} 4 & -1 & -2 & -1 & 0 \\ -1 & 4 & -1 & 0 & -1 \\ -2 & -1 & 4 & -1 & 0 \\ -1 & 0 & -1 & 4 & -1 \\ 0 & -1 & 0 & -1 & 4 \end{vmatrix} = d_n.$$

The dexter diagonal has for each element 4, viz. the sum of the four resistances, each to unity, which form each mesh or cell. And all the other figures, say, in the n th row, are the resistances (with minus sign prefixed) separating the n th mesh from all other meshes, zero being placed in the column corresponding to any mesh which has no common conductor or branch with this n th mesh. The order in which the columns stand and also the rows correspond to the order in which the meshes are numbered in fig. 7.

The numerical value of this determinant is easily found to be $288 = 3 \times 96 = d_n$. Now if we take the first minor of its leading element, we get a determinant formed of the elements included in the dotted rectangle; and taking this as a separate determinant and evaluating it, we have its value

$$d_{n-1} = 192 = 2 \times 96;$$

hence the resistance of the network between the points A and B is

$$\frac{d_n}{d_{n-1}} = \frac{288}{192} = 1\frac{1}{2} \text{ units.}$$

§ 9. One more simple numerical case may be taken and compared with the results of known methods.

Let a hexagon of conductors be taken (fig. 8) having crossed diagonals all meeting in the centre. Let the resistance of each side, as ab , be unity, and also let the resistance of each semidiagonal, as $0a$, be unity. Then required the combined resistance of this network of 12 conductors between the points A and B diametrically opposite. Join the points A and B by a dotted line of zero resistance, making an added mesh 1.

Mark the other meshes 2, 3, 4, 5, 6, 7. Then by forming the network equations it is easily seen that the network determinant d_n is

$$\begin{vmatrix} 3 & -1 & -1 & -1 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & -1 & 0 & 0 \\ -1 & -1 & 3 & -1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 3 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & -1 & 0 & -1 & 3 \end{vmatrix} = d_n.$$

The value of this determinant is 256.

The first minor of the leading element of d_n is d_{n-1} .

$$= \begin{vmatrix} 3 & -1 & 0 & -1 & 0 & 0 \\ -1 & 3 & -1 & 0 & 0 & 0 \\ 0 & -1 & 3 & 0 & 0 & -1 \\ -1 & 0 & 0 & 3 & -1 & 0 \\ 0 & 0 & 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & 0 & -1 & 3 \end{vmatrix}$$

The value of this last is 320.

Hence the resistance of the network between the points A and B is

$$R = \frac{d_n}{d_{n-1}} = \frac{256}{320} = \frac{4}{5}.$$

We can easily verify this result in the above symmetrical case, for the hexagonal framework in fig. 8 is traversed symmetrically by the current flowing through it; and hence no disturbance of the distribution of currents will take place by separating it, as in fig. 9. We break the connection between the semidiagonal conductors a, b and the mean diagonal A B, whilst keeping them in contact with each other, the resistance of each branch still remaining unity. It is then easily seen that the hexagon so arranged must offer exactly the same resistance between the points A and B as in its original form.

Now the combined resistance of a, b , and f , each equal to unity, between the points C, D is $\frac{3}{2}$; and the combined resistance of this with e and g in series is $2\frac{3}{4}$; and hence the total resistance of the whole network between A and B is equal to that of three conductors in multiple arc whose resistances are

respectively $2\frac{2}{3}$, 2, and $2\frac{2}{3}$, which is equal to

$$\frac{1}{\frac{1}{2\frac{2}{3}} + \frac{1}{2} + \frac{1}{2\frac{2}{3}}} = \frac{4}{5},$$

the same result as obtained above.

These numerical examples show conclusively that, in cases in which the resistance of a network can be obtained by simple direct methods, the results coincide, as should be the case, with those obtained by the employment of the general method; but at the same time the general method is capable of conducting easily to a solution in the most unsymmetrical cases. The general rule will, for instance, just as easily give the determinants when the selected points between which the resistance is required are not symmetrically placed, but are, say, adjacent angles of the hexagon, in which case no such simple direct method as employed above can be used.

§ 10. The following example will give a good illustration of Maxwell's method of treating network problems, viz. the case of Sir W. Thomson's resistance-balance for small resistances. In this arrangement (fig. 10) 9 conductors join 6 points and form 4 cells. B is the battery-circuit in which operates an electromotive force E. Let the four cycle currents be denoted by $x+y$, y , z , and w . These are the imaginary like-directed currents round the circuits, and the real currents in the branches are the differences of these.

The problem is to determine the current in the galvanometer branch G, and the relation of the resistances when this current through G is zero. Let P, Q, S, T, R, r , D be respectively the resistances of the branches, and G the resistance of the galvanometer circuit, and B the resistance of the battery circuit. Then $x+y$ and y being the imaginary like-directed currents in the two adjacent meshes of which the galvanometer branch is the common boundary, then $x+y-y=x$ is the current through the galvanometer.

Proceeding to write down the cycle equations, according to Maxwell's rule, we have

$$\begin{aligned}(P+G+Q+R)x+y-Gy-Qz-Rw &= 0, \\ (T+r+S+G)y-Gx+y-Sz-rw &= 0, \\ (Q+S+D)z-Sy-Qx+y-Dw &= 0, \\ (R+D+r+B)w-Rx+y-Dz-ry &= E.\end{aligned}$$

Rearranging these equations and solving for x , we have the following value :—

$$x = \frac{E}{\Delta} \begin{vmatrix} -\overline{Q+S}, & D, & -D \\ T+S+r, & T+r, & -r \\ P+Q+R, & P+R, & -R \end{vmatrix},$$

in which Δ is the determinant of the four equations in x , y , z , and w , and whose specific value does not concern us.

This gives the current in the galvanometer-branch; and if this is zero, then the determinant in the numerator of the equation giving x must be zero. Hence, when x is zero, we have

$$\begin{vmatrix} -\overline{Q+S}, & D, & 0 \\ T+S+r, & r+T, & T \\ P+Q+R, & P+R, & P \end{vmatrix} = 0,$$

this determinant being derived from the one in the equation for x by adding the second and third columns for a new third column.

This last determinant equation writes out into

$$(Q+S+D) \begin{vmatrix} T, & r \\ P, & R \end{vmatrix} + D \begin{vmatrix} T, & S \\ P, & Q \end{vmatrix} = 0.$$

Hence the condition that the current in the galvanometer-branch shall be zero is that both determinants in this expression shall be simultaneously zero, or

$$\begin{vmatrix} T, & r \\ P, & R \end{vmatrix} = 0, \quad \text{and} \quad \begin{vmatrix} T, & S \\ P, & Q \end{vmatrix} = 0;$$

that is,

$$\frac{T}{P} = \frac{r}{R} = \frac{S}{Q}.$$

Hence this condition expresses the relation which must hold good between the magnitudes of the resistances T , P , Q , S , r , R , in order that the galvanometer-branch G may be conjugate to the battery-branch B .

The above example shows well the symmetry of the method when dealing with a case of distribution of currents in a network.

§ 11. As a final illustration, let us consider the case of a circular wire $APBQ$, with a diametral wire PQ across it.

Take any two points A , B , at the extremities of a diameter

not coinciding with P Q, but separated by an angular distance θ from it, and let us obtain the resistance of the circular wire so crossed between the points A and B.

Join the points A, B by a dotted line of zero resistance. Call the three meshes so formed x , y , and z ; let r be the radius of the circle; and let ρ be the electrical resistance of the wire per unit of length. Then the

$$\text{Resistance of branch PQ} = 2\rho r,$$

$$\text{AP} = \rho r \theta,$$

$$\text{AQ} = \rho r (\pi - \theta),$$

and

$$\text{Resistance of branch BQ} = \text{resistance of AP},$$

$$\text{PB} = \text{AQ}.$$

Then the network determinant d_n is

$$\begin{vmatrix} \rho r \pi, & -\rho r (\pi - \theta), & -\theta \\ -\rho r (\pi - \theta), & \rho r (\pi + 2), & -\rho r 2 \\ -\theta, & -\rho r 2, & \rho r (\pi + 2) \end{vmatrix}.$$

Removing the common factor $r\rho$, we have to evaluate

$$\begin{vmatrix} \pi & -\pi - \theta & -\theta \\ -\pi - \theta & \pi + 2 & -2 \\ -\theta & 2 & \pi + 2 \end{vmatrix}.$$

This is very easily reduced to

$$\pi \begin{vmatrix} 0 & \theta & \pi \\ -1 & 1 & 2 \\ -\theta & 2 & \pi \end{vmatrix},$$

which is equal to

$$2\pi(\pi + \theta\pi - \theta^2),$$

and therefore

$$d_n = r^3 \rho^3 2\pi(\pi + \theta\pi - \theta^2).$$

The value of the minor of the leading element of the network determinant, viz. d_{n-1} , is

$$r^2 \rho^2 \{(\pi + 2)^2 - 4\} = r^2 \rho^2 \pi(\pi + 4).$$

Hence the resistance of the network between A and B, $= R$, is

$$\begin{aligned} \frac{d_n}{d_{n-1}} &= R = r\rho \frac{2\pi(\pi + \theta\pi - \theta^2)}{\pi(\pi + 4)} \\ &= r\rho \frac{2\pi + 2\pi\theta - 2\theta^2}{\pi + 4}. \end{aligned}$$

We can check this in the extreme cases when $\theta=0$ or $\theta=\frac{\pi}{2}$.

When $\theta=0$, the network-resistance is simply that of three conductors whose resistances are $2\rho r$, $\pi\rho r$, and $\pi\rho r$ joined in multiple arc, as in Plate IV. fig. 12, because PQ now coincides with AB; and this is simply $r\rho\frac{2\pi}{\pi+4}$. It is seen at once that the above value for R becomes this when θ is put equal to zero. Now, when $\theta=\frac{\pi}{2}$, the diameter PQ joins points at equal potential (fig. 13), and is not traversed by any current at all; and hence its removal will not affect the resistance between the points A and B.

Hence the resistance of the network simply reduces to that of a circle measured at the ends of a diameter, or to two conductors of resistance $\pi\rho r$ joined in multiple arc, and this is equal to $r\rho\frac{\pi}{2}$. By putting $\theta=\frac{\pi}{2}$ in the general solution

for R above, we get it reduced to $r\rho\frac{\pi}{2}$; and accordingly this formula agrees, as it should do in these reduced cases, with the results of the direct method based on first principles. If a value of θ be found which will make the expression

$$2\pi + 2\pi\theta - 2\theta^2$$

equal to $\pi + 4$, then for such a position of the diameter AB relatively to PQ the resistance of the circle and its diagonal PQ would be exactly equal to the resistance of half the diametral wire or to its radius, assuming both the circle and diagonal to be made of wire of equal conductivity per unit of length. To find the value of θ for which this is the case, we have to solve the quadratic

$$2\pi + 2\pi\theta - 2\theta^2 = \pi + 4.$$

If we put $\theta = \frac{\pi}{180} x^\circ$, where x° is the number of degrees equivalent to the angle θ , we find, as a solution for this quadratic, that the positive root is nearly

$$171^\circ 804.$$

Now 3 radians, or 3 unit-angles in circular measure, are nearly

$$171^\circ 887.$$

Hence, for a position of the diagonal PQ as in fig. 14, when the arc AP is nearly equal to $\pi - 3$, or to the fractional part of π , the resistance of the circle and diagonal PQ measured between the points A, B is very nearly equal to that of half the diagonal PQ; or, which is the same thing, the resistance of PQ alone is nearly double the combined resistance of the circle and diagonal measured between the points A and B at the extremity of a diameter removed $171^{\circ}804$ from PQ.

§ 12. A small practical application of this last example may be made in constructing a variable resistance.

Let PAQB (fig. 15) be a narrow circular canal cut in a slab of wood or ebonite and filled with mercury. Let PDQ be a bent copper wire balanced on a pivot CD, and having its ends P and Q dipping in the trough at opposite extremities of a diameter of the circular trough PAQB.

The total resistance between any two points A and B in the trough, which are also diametrically opposite, can be varied within limits by changing the position of PQ relatively to AB.

When PQ is turned so that it is at right angles to the diameter AB, it does not affect the total resistance between A and B, and may be removed. The resistance is then just that of the circular band of mercury taken at opposite extremities of its diameter. When PQ is coincident with AB it reduces the resistance, and in intermediate positions the joint resistance of trough and diagonal wire is intermediate between the greatest and least when it is in position removed either 90° or 0° from AB.

By using a circular glass canal filled with sulphate-of-zinc solution, and a zinc diagonal electrode and amalgamated-zinc electrodes at A and B, a variable resistance may be constructed capable of being varied over considerable ranges perfectly gradually and with no imperfect contacts.

§ 13. Having illustrated, by the foregoing examples, the methods of calculating both the currents in and resistances of networks of any complexity, we return for a moment to some general considerations.

Consider a function formed of the sum of each separate resistance in a network multiplied by the square of the current strength flowing through it. This expresses the heat generated per second in the whole network by that distribution of

current. This is called the Dissipation Function of the network. It represents the rate at which energy is being transformed into heat or rendered unavailable.

Write down the dissipation function for the network in fig. 1. Call it H . Then

$$H = Bx^2 + Ix - y^2 + Hx - r^2 + Cy^2 + Lz - y^2 + Az^2 + Ju - y^2 + Kz - w^2 + (D + E)u^2 + (F + G)w^2 + Mu - w^2.$$

Now the cycle equation for the cycle or mesh y is, by Maxwell's rule,

$$(C + I + L + J)y - Ix - Lz - Ju = 0,$$

which is the same as

$$Cy - Ix - y - Lz - y - Ju - y = 0.$$

And this is at once seen to be identically the same as the first partial differential of the dissipation function with respect to the cyclic symbol y , or is the same as

$$\frac{1}{2} \frac{\partial H}{\partial y} = 0,$$

where ∂ represents partial differentiation; and by writing down the other cycle equations for each cyclic symbol or imaginary current, x , y , z , &c., we can show that these current-equations are respectively

$$\frac{1}{2} \frac{\partial H}{\partial x}, \quad \frac{1}{2} \frac{\partial H}{\partial y}, \quad \frac{1}{2} \frac{\partial H}{\partial z}, \quad \&c.,$$

each equated to the effective electromotive force in that cycle or mesh.

Let us assume now that x is constant, but that y , z , u , w , &c. are independent variables and are arbitrarily changed. This is equivalent to supposing that a given quantity of electricity per second is pushed into the network, but that its distribution is supposed to be varied. We see that the equations which we write down, according to Maxwell, to determine the *real* distribution of currents in the network, according to Ohm's law, are the same equations as would be written down to find the values of y , z , u , w , &c., which make the dissipation function a minimum under fixed conditions of total current flowing into the network, viz. equating to zero the first partial differentials of H with respect to the variables y , z , u , &c. The same holds good generally, hence we see that this is

another way of arriving at the theorem of which Maxwell has given a proof on page 375, § 284, vol. i. of his large Treatise, 2nd edition, viz.:—"In any system of conductors in which there are no internal electromotive forces the heat generated by currents distributed in accordance with Ohm's law is less than if the currents had been distributed in any other manner consistent with the actual conditions of supply and outflow of the current."

The exact proof that the partial differentials of the dissipation function equated to zero gives the condition that the dissipation function shall be a *minimum* is not complete without an examination of Lagrange's conditions. It is obvious that the second partial differentials of the dissipation function are quantities which are resistances, viz. the coefficients of the current symbols in the cycle equations, and that the conditions for a minimum are complied with, since $\frac{\partial^2 H}{\partial y^2}$, &c. are positive; and the *discriminant* of the quadratic function of the currents or symmetrical determinants formed of these second partial differentials is what has been called above the network determinant. This and all its successive minors are positive quantities*.

§ 14. In the foregoing sections the problems have been treated under the limitations that the various meshes of the network of conductors have no mutual and no self-induction. The introduction of these inductive actions will affect in a considerable way the treatment of the problem; and the distribution of the currents in, and the resistance of, the network will be affected by them during the time taken by the currents to become steady.

In those pages of his Treatise in which Clerk Maxwell worked out his splendid dynamical theory of electromagnetism, he starts with the explanation of the methods Lagrange and Hamilton employed to bring pure dynamics under the power of analysis, and the results of Lagrange are embodied in the equation

$$X = \frac{d}{dt} \frac{dT}{dx} - \frac{dT}{dx},$$

* See Williamson's 'Differential Calculus,' p. 408, "On the Conditions for a Maximum and Minimum of a Function of any number of Variables," § 163, and Appendix.

in which X is the impressed force tending to increase the variable x , and T denotes the visible energy of the system of bodies at that instant.

This equation establishes a relation between the kinetic energy of a material system at any instant, the force impressed upon it in a certain direction, and a quantity called a variable, which expresses the state or condition of the system with respect to that direction. Maxwell, by a process of extraordinary ingenuity, extended this reasoning from motive forces, masses, velocities, and kinetic energies of gross matter to the electromotive forces, quantities, currents, and electrokinetic energies of electrical matter, and in so doing obtained a similar equation of great generality for attacking electrical problems.

In the electrical problem the variables are the quantities of electricity x, y, z , &c. which have from the beginning of the epoch flowed past any points, and the analogues of the velocities are the fluxes of these, $\dot{x}, \dot{y}, \dot{z}$, &c., or the currents.

The electrokinetic energy is measured by the quadratic expression

$$T = \frac{1}{2} L_1 \dot{x}_1^2 + \frac{1}{2} L_2 \dot{x}_2^2 + \dots + M_{12} \dot{x}_1 \dot{x}_2 + \dots, \text{ \&c.},$$

where the coefficients L_1, L_2, M_{12} are functions of the geometrical variables, but into which the electrical variables do not enter.

If now, as before, $\dot{x}_1, \dot{x}_2$ represent the imaginary like-directed currents round each mesh of a network, in which currents are beginning to flow, then

$$\frac{dT}{dx_1} \text{ and } \frac{dT}{dx_2}, \text{ \&c.}$$

represent the electrokinetic momenta of these circuits. Denote them by p_1, p_2 , &c., and accordingly

$$p_1 = L_1 \dot{x}_1 + M_{12} \dot{x}_2, \text{ \&c.}$$

If E is the impressed electromotive force in the circuit or mesh arising from some cause, battery, thermopile, dynamo machine, &c., which would produce a current independently of magneto-induction, then, if R be the total resistance round the mesh, and $\dot{x}$ the cyclic current, $R\dot{x}$ is the electromotive force required to overcome the resistance of the circuit,

and $E - R\dot{x}$ is the electromotive force available for changing the electric momentum of the circuit.

Accordingly, by Lagrange's equation,

$$E - R\dot{x} = \frac{dp}{dt} - \frac{dT}{dx},$$

where T is the electrokinetic energy. As T does not contain x , that is to say it is a function of currents, not quantities, the last term disappears, and we have

$$E - R\dot{x} = \frac{d}{dt} \frac{dT}{dx},$$

or

$$\frac{d}{dt} \frac{dT}{dx} + R\dot{x} = E.$$

The electromotive force is therefore expended in two things: first, overcoming the resistance R ; and, secondly, increasing the electromagnetic momentum p . Now if there is no electromagnetic momentum, we have seen that the cyclic equations are of the form

$$\frac{1}{2} \frac{dH}{dx} = E',$$

where H is the dissipation function of the system, and E' is the acting electromotive force concerned in overcoming the resistance of the circuit.

If, then, we substitute for $R\dot{x}$ in equation $\frac{1}{2} \frac{dH}{dx}$, we have as the general equation for the electromotive force in any mesh or cycle x ,

$$\frac{d}{dt} \frac{dT}{dx} + \frac{1}{2} \frac{dH}{dx} = E.$$

This most important equation is Maxwell's general equation for determining the current x in any circuit when the dissipation function, and kinetic energy, and impressed electromotive force are known. We shall proceed to apply it to the solution of some network problems, in which the self and mutual induction of the branches is taken into account to determine the distribution of currents and combined resistance at any instant during the variable state.

§ 15. Consider, first, the case of a galvanometer with a coefficient of self-induction L and resistance G , and shunted by a shunt of resistance S , but wound so as to have no coefficient of self-induction, and let the shunt and galvanometer-coils be so far removed that there is no coefficient of mutual induction. This is the ordinary practical case.

Let a battery be joined up and let the battery and connections have a resistance B and electromotive force E (see fig. 16).

We have then a two-mesh network. Call the current in the galvanometer- and shunt mesh y and the current in the shunt and battery mesh $x+y$. Then the current through the galvanometer is y , the current through the shunt is x , and the current through the battery is $x+y$.

The dissipation function Π is

$$B\overline{x+y^2} + S\overline{x^2} + G\overline{y^2} = H,$$

which may be written

$$\overline{B+S} \overline{x+y^2} + \overline{G+S} \overline{y^2} - 2S \overline{x+y} \overline{y} = H;$$

and the electromagnetic energy is

$$\frac{1}{2} L \overline{y^2} = T.$$

Hence, by the general equation,

$$\frac{d}{dt} \frac{dT}{dy} + \frac{1}{2} \frac{dH}{dy} = E,$$

we have the two cycle equations for the y and $x+y$ cycles,

$$\left. \begin{aligned} \frac{d}{dt} L y + \overline{G+S} y - S \overline{x+y} &= 0 \\ \overline{B+S} \overline{x+y} - S y &= E, \end{aligned} \right\}$$

and

or

$$\left. \begin{aligned} \left(L \frac{d}{dt} + G \right) y - S x &= 0, \\ B y + \overline{B+S} x &= E. \end{aligned} \right\}$$

and

The solution of these for x and y is

$$y = \frac{\begin{vmatrix} E & B+S \\ 0 & -S \end{vmatrix}}{\begin{vmatrix} B & B+S \\ L\frac{d}{dt}+G & -S \end{vmatrix}} = \text{current through [galvanometer,}$$

and

$$x = \frac{\begin{vmatrix} B & E \\ L\frac{d}{dt}+G & 0 \end{vmatrix}}{\begin{vmatrix} B & B+S \\ L\frac{d}{dt}+G & -S \end{vmatrix}} = \text{current through [the shunt.}$$

Writing out this differential equation for y' we have,

$$\begin{vmatrix} L\frac{d}{dt} & -G \\ 0 & B+S \end{vmatrix} y + \begin{vmatrix} B & B+S \\ -G & S \end{vmatrix} y = \begin{vmatrix} E & -B+S \\ 0 & S \end{vmatrix},$$

$$\text{or } \overline{B+S} L \frac{dy}{dt} + (BS + RG + SG)y = ES,$$

$$\text{or } \frac{dy}{dt} + \frac{BS + BG + SG}{(B+S)L} y = \frac{ES}{(B+S)L}.$$

The solution of this differential equation is

$$y = \frac{ES}{BG + GS + BS} \left(1 - e^{-\frac{BG + GS + BS}{(B+S)L} t} \right).$$

This gives the value of the current through the galvanometer at any time, t , after starting the flow by making the connection with the battery.

When $t=0$, then $y=0$, and as t increases y increases, and finally, when $t=\infty$, $y = \frac{ES}{BG + GS + BS}$, or, as it may be written, $y = \frac{S}{G+S} \frac{E}{B + \frac{SG}{G+S}}$.

This last is the ordinary formula given for the current through a shunted galvanometer; but we see that when self-induction is taken into account, it is not until after an infinite time that the current rises to this value.

By the cycle equation, $By + \overline{B+S}x = E$: hence

$$x = \frac{E - By}{B + S}.$$

And if we write N for the factor $\left(1 - e^{-\frac{BG + GS + BG}{(B+S)L}t}\right)$, then

$$\begin{aligned} x &= \frac{E - \frac{EBSN}{BG + BS + SG}}{B + S} \\ &= \frac{E(BG + BS + SG - BSN)}{(B + S)(BG + BS + SG)}; \end{aligned}$$

which gives the current through the shunt at any instant.

§ 16. Consider now the combined resistance of the galvanometer and shunt at any instant.

The self-induction of the galvanometer acts like a spurious resistance during the commencement of the current and drags out or prolongs the rise of current in the galvanometer-coils ; accordingly, during this period the combined resistance is a function of the time t from the commencement of the flow.

To calculate the combined resistance of galvanometer and shunt at any instant, we proceed as in the cases above exemplified. Form the cycle equations

$$\begin{aligned} (B + S)\overline{x+y} - Sy &= E, \\ -S\overline{x+y} + \left(G + S + L\frac{d}{dt}\right)y &= 0. \end{aligned}$$

Write down the determinant of these equations with the battery-circuit resistance put equal to zero, that is put $B=0$, and the combined resistance R required is the quotient of this determinant by its first minor, viz.

$$R = \frac{\begin{vmatrix} S & -S \\ -S & G + S + L\frac{d}{dt} \end{vmatrix}}{G + S + L\frac{d}{dt}}$$

or

$$R = \frac{\begin{vmatrix} S & 0 \\ -S & G + L\frac{d}{dt} \end{vmatrix}}{S + G + L\frac{d}{dt}} = \frac{S\left(G + L\frac{d}{dt}\right)}{S + \left(G + L\frac{d}{dt}\right)}.$$

We have now to see what is the meaning of $G + L \frac{d}{dt}$ as an operator in a determinant.

If we consider the formation of a current in a circuit of resistance R and coefficient of self-induction L by an electromotive force E , we have the equation for the current i

$$L \frac{di}{dt} + Ri = E.$$

Write thus

$$\left(L \frac{d}{dt} + R \right) i = E;$$

or, by notation of the calculus of operations,

$$i = E \left(L \frac{d}{dt} + R \right)^{-1}.$$

But now the solution of the above differential equation under the conditions $t=0$, $i=0$, and $t=\infty$, $i=\frac{E}{R}$, is

$$i = E \frac{\left(1 - e^{-\frac{R}{L}t} \right)}{R}.$$

Comparing these two expressions for i together, we have

$$L \frac{d}{dt} + R = \frac{R}{\left(1 - e^{-\frac{R}{L}t} \right)}.$$

Hence we may substitute in the expression for the combined resistance of galvanometer and shunt for $L \frac{d}{dt} + G$,

$$\frac{G}{\left(1 - e^{-\frac{G}{L}t} \right)},$$

and we have as a result,

$$R = \frac{SG}{G + S \left(1 - e^{-\frac{G}{L}t} \right)}.$$

We see that when $t=0$, $R=S$, and when $t=\infty$, $R=\frac{GS}{G+S}$.

Hence the result shows that at the first instant of starting a current through a shunted galvanometer, when the shunt has no self-induction and the galvanometer a considerable one, the galvanometer behaves as if it had a high spurious resistance, which in time dies away, allowing the total current,

after an infinite time, to be divided between the galvanometer and the shunt in the ratio of $\frac{S}{G+S}$ to $\frac{G}{G+S}$.

§ 17. We may apply the same methods to the examination of the case when the current sent through the shunted galvanometer is not generated by a source of constant electromotive force, but is a discharge from a condenser.

Let K (fig. 17) be a condenser connected up with a shunted galvanometer, so that when the key k is pressed a discharge passes through the galvanometer and shunt. Call the two cycles x and y . Let G be the galvanometer-resistance and S the shunt, and let L_1 and L_2 be their respective coefficients of self-induction; the coefficient of mutual induction being zero.

Let q be the quantity of electricity in the condenser at any instant t . Counting the time from the instant of commencing the discharge, let C be the capacity of the condenser, and let q_1 and q_2 be the quantities of electricity which have, since the beginning of the epoch, flowed respectively through the galvanometer and the shunt.

If T be the energy function and F the dissipation function, we have, as above, the fundamental equations

$$2T = L_1 y^2 + L_2 (x - y)^2,$$

and

$$2F = Gy^2 + S(x - y)^2;$$

or

$$2T = L_1 y^2 + L_2 x^2 + L_2 y^2 - 2L_2 xy,$$

$$2F = Gy^2 + Sx^2 + Sy^2 - 2Sxy.$$

By the fundamental equation

$$\frac{d}{dt} \frac{dT}{dx} + \frac{dF}{dx} = e.$$

For e we must write $\frac{q}{C}$.

Writing, then, the cycle equations, we have

$$\frac{d}{dt} (L_2 x - L_2 y) + Sx - Sy = \frac{q}{C}.$$

$$\frac{d}{dt} (L_1 y + L_2 y - L_2 x) + Gy + Sy - Sx = 0;$$

from which we deduce easily

$$L_1 \frac{dy}{dt} + Gy = \frac{q}{C}$$

and

$$L_2 \frac{d}{dt}(x-y) + S(x-y) = \frac{q}{C};$$

or

$$L_1 \frac{dy}{dt} - L_2 \frac{d}{dt}(x-y) = S(x-y) - Gy.$$

Now y and $x-y$ represent the strengths of the currents flowing through the galvanometer and the shunt at any instant.

If we integrate both sides of the equation from 0 to ∞ , we have

$$[L_1 y - L_2 (x-y)]_0^\infty = S \int_0^\infty (x-y) dt - G \int_0^\infty y dt.$$

Now the left-hand side of the equation is zero because quantities of the form of Ly represent the number of lines of force which are added into the circuit of the galvanometer, and the discharge may be divided into two parts, during one of which lines of force are being added to, and in the other of which subtracted from, the circuits of the galvanometer and shunt; and

the sum of these is zero. Again, $\int_0^\infty (x-y) dt$ and $\int_0^\infty y dt$ represent the whole quantities q_2 and q_1 of electricity which have flowed respectively through the galvanometer and the shunt. Hence we arrive at the conclusion that

$$Sq_2 - Gq_1 = 0,$$

or

$$\frac{G}{S} = \frac{q_2}{q_1};$$

that is, the total quantity of the discharge is divided between the two circuits inversely as their resistances. We see therefore that self-induction does not affect the ratio of division of a discharge in a divided circuit, provided that no external work, such as the moving of magnets or circuits conveying currents, absorbs current energy. Hence, if a ballistic galvanometer is shunted and a discharge sent through it, if the needle has sufficient moment of inertia and the discharge is sufficiently short, so that the needle has not perceptibly moved from its position before the discharge is over, then the whole quantity of electricity is divided between the galvanometer and the shunt in the inverse ratio of their resistances.

§ 18. To complete the solution we have to calculate the current flowing through the galvanometer and shunt at any instant.

Taking the two cycle equations

$$L_1 \frac{dy}{dt} + Gy = \frac{q}{C}, \quad \dots \dots \dots (i.)$$

and

$$L_2 \frac{d}{dt} \overline{x-y} + S \overline{x-y} = \frac{q}{C}, \quad \dots \dots \dots (ii.)$$

we get

$$L_1 \frac{dy}{dt} = \frac{q}{C} - Gy,$$

and

$$\begin{aligned} L_1 L_2 \frac{dx}{dt} &= (L_1 + L_2) \frac{q}{C} - L_1 S \overline{x-y} - L_2 Gy \\ &= (L_1 + L_2) \frac{q}{C} - L_1 Sx + (L_1 S - L_2 G)y. \quad \dots (iii.) \end{aligned}$$

Differentiate this last equation with regard to t and eliminate $\frac{dy}{dt}$ by the help of the equation above it, and we arrive at

$$L_1 L_2 \frac{d^2 x}{dt^2} = -L_1 S \frac{dx}{dt} + \frac{L_1 S - L_2 G}{L_1} \left(\frac{q}{C} - Gy \right).$$

Eliminating y between the last and equation (iii.) and reducing, we arrive at

$$L_1 L_2 \frac{d^2 x}{dt^2} + (L_1 S + L_2 G) \frac{dx}{dt} + GSx = (G + S) \frac{q}{C};$$

but now $x = -\frac{dq}{dt}$. Making this substitution we have

$$L_1 L_2 C \frac{d^2 q}{dt^2} + C (L_1 S + L_2 G) \frac{dq}{dt} + CGS \frac{dq}{dt} + (G + S)q = 0, \quad (iv.)$$

an interesting equation, the solution of which gives us the quantity of electricity in the condenser at any instant, t , after starting the discharge. According to the equation above

$$L_1 \frac{dy}{dt} + Gy = \frac{q}{C}.$$

This equation gives us a value of y or the current through the galvanometer at any instant when we know q , or the

quantity left in the condenser at that instant. The above may be written

$$y = \left(L_1 \frac{d}{dt} + G \right)^{-1} \frac{q}{C},$$

and the final equation (iv.) may be written

$$\frac{q}{C} = \left(L_1 L_2 C^2 \frac{d^3}{dt^3} + (L_1 S + L_2 G) C^2 \frac{d^2}{dt^2} + C^2 G S \frac{d}{dt} + C(G + S) \right)^{-1} 0;$$

and accordingly we have the following equation for the value of y at any instant

$$y = \left(L_1 \frac{d}{dt} + G \right)^{-1} \left(L_1 L_2 C^2 \frac{d^3}{dt^3} + (L_1 S + L_2 G) C^2 \frac{d^2}{dt^2} + C^2 G S \frac{d}{dt} + C(G + S) \right)^{-1} 0,$$

which may be written

$$y = \left(L_1^2 L_2 C^2 \frac{d^4}{dt^4} + (L_1^2 S C^2 + 2 L_1 L_2 C^2 G) \frac{d^3}{dt^3} + (2 L_1 C^2 G S + L_2 G^2 C^2) \frac{d^2}{dt^2} + (G + S C L_1 + C^2 G^2 S) \frac{d}{dt} + G C (G + S) \right)^{-1} 0.$$

This linear differential equation is solved when we know the roots of the auxiliary biquadratic; and according as they are all real or partly imaginary, so will be the nature of the solution.

If the roots are all real the solution is a sum of exponentials, whose total value first increases and then dies away as t increases, indicating that the discharge produces a wave of electricity through the galvanometer always in one direction; but if two or all of the roots of the auxiliary biquadratic are unreal, it indicates as the form of solution a function of sines and cosines which will have periodic values, and points to the fact that the discharge is a series of alternations. The general case, when both the galvanometer and shunt have coefficients of self-induction, when treated to determine the conditions for an oscillating discharge, leads to an expression of considerable complexity and not much practical use. The reduced case, in which the galvanometer is wound to have self-induction and the shunt so as to have no coefficient of self-induction, is, however, a practical case, and can be treated without much difficulty.

Taking the differential equation for q , equation (iv.), and

writing in it $L_2=0$, we have

$$CL_1S \frac{d^2q}{dt^2} + CGS \frac{dq}{dt} + (G+S)q=0,$$

or

$$\frac{S}{G+S} \frac{d^2q}{dt^2} + \frac{GS}{L_1(G+S)} \frac{dq}{dt} + \frac{q}{CL_1} = 0.$$

The discharge will be oscillatory if the auxiliary quadratic

$$\frac{S}{G+S} m^2 + \frac{GS}{L_1(G+S)} m + \frac{1}{CL_1} = 0$$

has unreal or imaginary roots.

Solving it we have

$$m^2 + \frac{Gm}{L_1} + \frac{G^2}{4L_1^2} = \frac{G^2}{4L_1^2} - \frac{G+S}{SCL_1},$$

or

$$\left(m + \frac{G}{2L_1}\right) = \pm \frac{\sqrt{G^2S^2C^2 - 4G + SL_1SC}}{2L_1SC}.$$

Hence, for the roots to be imaginary,

$$4L_1 \overline{G+S} SC \text{ must be greater than } G^2S^2C^2,$$

or

$$\frac{4L_1}{C} > \frac{G \cdot GS}{G+S}.$$

If this relation holds good, then the discharge is oscillatory in the condenser; and accordingly we see that to prevent electrical oscillation in the galvanometer circuit, the product of resistance of the galvanometer and combined resistance of galvanometer and shunt must be equal to or greater than four times the self-induction of the galvanometer divided by the capacity of the condenser.

We may write the solution of the quadratic above,

$$\begin{aligned} m &= -\frac{G}{2L_1} \pm \sqrt{-1} \sqrt{\frac{G+S}{L_1C} - \frac{G^2}{4L_1^2}} \\ &= -a \pm \sqrt{-1}\beta, \end{aligned}$$

where

$$a = \frac{G}{2L_1} \quad \text{and} \quad \beta = \sqrt{\frac{G+S}{L_1C} - \frac{G^2}{4L_1^2}};$$

and accordingly when β is real, that is when

$$\frac{\frac{G+S}{S}}{L_1 C} \text{ is } > \frac{G^2}{4L_1^2},$$

we have, for solution of equation,

$$q = Ae^{-at} \cos \beta t + \beta e^{-at} \sin \beta t.$$

When $t=0$, $q=Q$ =the original charge of the condenser, and

$$\frac{dq}{dt}=0 \text{ when } t=0;$$

therefore

$$Q=A \quad \text{and} \quad Q\frac{a}{\beta}=B;$$

and

$$q = Qe^{-at} \left(\cos \beta t + \frac{a}{\beta} \sin \beta t \right).$$

Having now the value of the quantity of electricity left in the condenser at any instant, we can find easily, from the cycle equation (i.), the value of the current through the galvanometer. For

$$L_1 \frac{dy}{dt} + Gy = \frac{q}{C},$$

or

$$\frac{dy}{dt} + \frac{G}{L} y = \frac{q}{CL};$$

$$\therefore y = e^{-\int \frac{G}{L} dt} \left\{ C' + \int e^{\int \frac{G}{L} dt} \frac{q}{CL} dt \right\},$$

and the constant C' is determined by the condition $y=0$ when $t=0$.

Substituting the value of q above, we have

$$a = \frac{G}{2L},$$

$$y = e^{-2at} \left\{ C' + \frac{1}{\beta} \frac{Q}{CL} \int e^{at} (\beta \cos \beta t + a \sin \beta t) \right\}$$

$$= e^{-2at} \left\{ C' + \frac{Q}{\beta CL} e^{at} \sin \beta t \right\};$$

but $C'=0$,

$$\therefore y = \frac{Q}{\beta CL} e^{-at} \sin \beta t;$$

and since

$$a = \frac{G}{2L_1} \quad \text{and} \quad \beta = \sqrt{\frac{G+S}{L_1 C} - \frac{G^2}{4L_1^2}},$$

$$y = \frac{Q}{\sqrt{\frac{G+S}{S} CL_1 - \frac{G^2 C^2}{4}}} e^{-\frac{G}{2L_1} t} \sin \left(\sqrt{\frac{G+S}{SL_1 C} - \frac{G^2}{4L_1^2}} t \right),$$

which gives the value of the instantaneous current in the galvanometer-circuit at any instant t after starting a discharge from a condenser of capacity C and original quantity Q through a shunted galvanometer, the shunt being wound without self-induction, and the galvanometer having a coefficient of self-induction L_1 .

§ 19. Two concluding examples of this method of treating network problems will now be given, which are in Professor Clerk Maxwell's own words*.

Theorem.—To compare the induction between one pair of coils and any other two.

Let $\alpha, \beta, \gamma, \delta$ be four coils of wire.

It is required to compare the mutual induction of α and γ with that of β and δ .

Join up α and β coils in series with a galvanometer, and join up γ and δ coils in multiple arc with a battery, as shown in fig. 18.

Place the coils in position.

* In the May term 1879, Professor Clerk Maxwell lectured at Cambridge on Electromagnetism, and in the two last lectures of the Course he gave this method of obtaining the equation for the currents in a network of conduction. In the last lecture of all he applied the method to cases in which self and mutual induction was taken into account, and gave the two illustrations in § 19. At the conclusion of this lecture he had ended his professorial duties for the term, and a melancholy interest attaches to the subject which occupied his mind on the last occasion on which, unconsciously to himself or his pupils, he was to perform them. Those who enjoyed even for a brief period the privilege of being taught by him, ever cherish a vivid remembrance of the intellectual treat afforded by Professor Maxwell's lecture-teaching, and the profound suggestiveness and interest of it.

The two examples in § 19 and § 20 are taken from my notes of Prof. Maxwell's lectures, with some little alterations to make them clearer.

Let R and S be resistances of the primaries γ and δ ; let Q be resistance of the two secondaries and of the galvanometer.

Let L_1, L_2, N_1, N_2 be the coefficients of self-induction of the coils, and M_1, M_2 the coefficients of mutual induction of a and γ, β and δ . Let Γ be the coefficient of self-induction of the galvanometer.

Call x the cycle current of γ, y that of δ , and z that of the circuit formed of a, β , and the galvanometer.

The kinetic energy T of the system is

$$2T = x^2 N_1 + 2xz M_1 + z^2 (L_1 + L_2 + \Gamma) + y^2 N_2 + 2yz M_2,$$

and the dissipation function F is

$$2F = x^2 R + y^2 S + z^2 Q.$$

Then, by the formula

$$E = \frac{d}{dt} \frac{dT}{dx} + \frac{dF}{dx},$$

$$\dot{x} N_1 + \dot{z} M_1 + x R = E.$$

$$\dot{y} N_2 + \dot{z} M_2 + y S = -E.$$

$$\dot{x} M_1 + \dot{y} M_2 + \dot{z} (L_1 + L_2 + \Gamma) + z Q = 0.$$

Now $x - y$ is the current through the battery; hence if we put $x + y$ for x in the above, we shall get x as the battery-current. Hence, making the change, we have

$$(\dot{x} + \dot{y}) N_1 + \dot{z} M_1 + \overline{x + y} R = E, \quad \dots \dots \dots (i.)$$

$$\dot{y} N_2 + \dot{z} M_2 + y S = -E, \quad \dots \dots \dots (ii.)$$

$$(\dot{x} + \dot{y}) M_1 + \dot{y} M_2 + \dot{z} (L_1 + L_2 + \Gamma) + z Q = 0; \quad \dots (iii.)$$

add equations (i.) and (ii.) and arrange, putting n for $\frac{d}{dt}$,

$$(N_1 n + R)x + (N_1 n + N_2 n + R + S)y + (M_1 + M_2)nz = 0,$$

$$M_1 nx + (M_1 n + M_2 n)y + \{ (L_1 + L_2 + \Gamma)n + Q \} z = 0.$$

Eliminating y , we have

$$\{ n(N_1 n + R)(M_1 + M_2) - M_1 n(N_1 n + R + S) \} x$$

$$+ \{ (M_1 + M_2)^2 n^2 - \{ (L_1 + L_2 + \Gamma)n + Q \} \{ N_1 n + N_2 n + R + S \} \} z = 0.$$

Hence we get

$$z = \frac{\{ n^2 (M_2 N_1 - M_1 N_2) - n (M_1 S - M_2 R) \} x}{\text{a denominator which does not concern us}}.$$

If matters are so arranged that $z=0$, or the galvanometer shows no current,

$$(M_2N_1 - M_1N_2) \frac{dx}{dt} = (M_1S - M_2R);$$

hence if there is no "kick" on the galvanometer on making the current, then

$$\frac{M_1}{M_2} = \frac{R}{S}.$$

§ 20. *Theorem.*—To determine the capacity of a condenser by means of a Wheatstone's bridge (fig. 19).

Let $\alpha, \beta, \gamma, \delta$ be the four points of a Wheatstone's bridge; and let the branch between α and β be interrupted at $a b$, and a Leyden jar or condenser inserted provided with some rapid commutator, such as a tuning-fork, so that whilst the outside of the jar is kept permanently attached to β , the inside is alternately joined to a and b .

If a tuning-fork is used and its prongs have small metal styles which just come down to the surface of the mercury in two little cups, when the fork vibrates, as the prongs come together, the upper point dips in; and as they separate, the lower one dips in; hence the shank of the fork is alternately connected with one and the other cup. The interval between the time of connection being exactly half the time of a complete oscillation of the fork.

Now let the meshes of the network be called $x+z, z$, and y ; then x is the current through the galvanometer, and y is the current through the battery. When the arrangement is made as in the diagram, and the fork set vibrating, the vibrating fork and the condenser act together like a resistance, and let through so much electricity per second.

Now, as the condenser gets its charge by electricity flowing into it, it builds up an opposing electromotive force in the z circuit which at any instant is equal to the value of $\int \frac{z dt}{K}$, where

K is the capacity of the jar, the integral being integrated from the instant when the charging commences up to the instant considered. Now, if the fork makes n vibrations a second when the steady state is set up, the current z which flows into

the jar has a mean value z ; and therefore $\frac{z}{nK}$ is the opposing electromotive force in that branch.

Accordingly, the condenser and associated commutator behave like a voltmeter inserted in the branch $\alpha\beta$, or like a resistance with a counter electromotive force in it. Only such a combined jar and fork differs from an ordinary metallic resistance in this, that its apparent resistance is not constant, but depends on two things, the speed of commutation or charge and recharge, and the capacity of the condenser; whilst the counter electromotive force depends on the current z , and, being represented by $\frac{z}{nK}$, is dependent not only on n and K , but also on the values of all the other resistances in the branches. In the first place, we require an expression for the electromotive force charging the condenser. Let the difference of potential between a and b be called e . Then consider the network formed by the five conductors R, S, Q, G , and B with the electromotive force in the branch B ; write down the network equations for this z mesh network.

$$\begin{aligned}(B + R + S)y - S(x + z) &= E, \\ -Sy + (Q + S + G)(x + z) &= 0.\end{aligned}$$

Hence

$$y = \frac{E(Q + S + G)}{\delta};$$

and

$$x + z = \frac{ES}{\delta},$$

where δ = the determinant

$$\begin{vmatrix} B + R + S, & -S \\ -S, & Q + S + G \end{vmatrix}$$

which is

$$S(Q + G) + (R + B)(Q + S + G).$$

Now the difference of potential e between a and b when the condenser is just beginning to be charged is

$$G(x + z) + Ry = e;$$

$$\therefore e = \frac{ESG}{\delta} + \frac{ER(Q+S+G)}{\delta}$$

$$= E \frac{SG + R(Q+S+G)}{S(Q+G) + (R+B)(Q+S+G)} = \frac{\begin{vmatrix} -G, & -R \\ Q+S+G, & -S \end{vmatrix}}{S(Q+G) + (R+B)(Q+S+G)},$$

or

$$e = E \frac{\begin{vmatrix} -G, & -R \\ Q+S+G, & -S \end{vmatrix}}{\delta}.$$

Now if the electromotive force e be employed n times in a second to charge a jar of capacity K , the average current flowing into the jar is $nKe = z$.

Now to find z we have to consider the distribution of currents when the fork or commutator is in operation, and the condenser allowing a flow of electricity to take place through it.

Let P be the resistance which could equivalently replace the jar and fork—that is, would allow an equal quantity of electricity to pass per second; then, since $\frac{z}{nK}$ is the opposing electromotive force in this branch, we have the following equation for the three cycles x , $x+z$, and y :—

$$-Sx + (R+S+B)y - (R+S)z = E,$$

$$-Gx - Ry + (P+R)z = -\frac{z}{nK},$$

$$(Q+S+G)x - Sy + (Q+S)z = 0.$$

Now let Δ stand for the determinant

$$\begin{vmatrix} -S, & R+S+B, & -(R+S) \\ -G, & -R, & P + \frac{1}{nK} + R \\ Q+S+G, & -S, & Q+S \end{vmatrix}.$$

Then the solution of the above equations for z and x are

$$z = \frac{E \begin{vmatrix} -G, & -R \\ Q+S+G, & -S \end{vmatrix}}{\Delta},$$

and

$$x = \frac{E \begin{vmatrix} -R, & P + \frac{1}{nK} + R \\ -S, & Q+S \end{vmatrix}}{\Delta}.$$

z is the average current flowing through the condenser, and x is the current through the galvanometer. Now let the resistances R , S , and Q be so varied that the current through the galvanometer is zero, then $x=0$; and therefore

$$\begin{vmatrix} -R, & P + \frac{1}{nK} + R \\ -S, & Q + S \end{vmatrix} = 0,$$

or

$$R(Q+S) = S\left(P + \frac{1}{nK} + R\right),$$

or

$$\frac{RQ}{S} = P + \frac{1}{nK}.$$

Now insert this value for $P + \frac{1}{nK}$ in the determinant Δ above and calculate its value, and we arrive at the expression

$$\Delta = \frac{\{B(Q+S) + Q(R+S)\}\{G(R+S) + R(Q+S)\}}{S}.$$

We have now, by substitution of this value of Δ in the value obtained above for z , an expression for the value of the average current through the condenser when the bridge is balanced, and it is

$$z = \frac{ES \begin{vmatrix} -G, & -R \\ Q+S+G, & -S \end{vmatrix}}{\{B(Q+S) + Q(R+S)\}\{G(R+S) + R(Q+S)\}}.$$

Equating this to the other value for z , namely,

$$z = nKe = nKE \frac{\begin{vmatrix} -G, & -R \\ Q+S+G, & -S \end{vmatrix}}{S(Q+G) + (R+B)\{Q+S+G\}},$$

we have

$$nK = \frac{S\{S(Q+G) + (R+B)(Q+S+G)\}}{\{B(Q+S) + Q(R+S)\}\{G(R+S) + R(Q+S)\}},$$

which gives us a value for nK in terms of

$$B, Q, R, S, G.$$

Now it is interesting to note that we may otherwise write the above expression for nK ,

$$\frac{1}{nK} = \frac{\Delta}{\delta},$$

where Δ is the determinant,

$$\begin{vmatrix} R+B+S, & -S, & -\overline{R+S} \\ -S, & Q+S+G, & R\left(\frac{Q}{S}+1\right) \\ -R, & -G, & Q+S \end{vmatrix},$$

and δ is its first minor,

$$\begin{vmatrix} R+B+S, & -S \\ -S, & Q+S+G \end{vmatrix},$$

and $\frac{1}{nK}$ is of the dimensions of a resistance.

The value for nK writes out by a simple transformation into another form,

$$nK = \frac{S \left\{ 1 - \frac{S^2}{(Q+S+G)(R+B+S)} \right\}}{RQ \left\{ 1 + \frac{SB}{Q(R+B+S)} \right\} \left\{ 1 + \frac{SG}{R(Q+S+G)} \right\}};$$

which is the form in which it is given by Prof. J. J. Thomson in his paper, and quoted by Mr. R. T. Glazebrook in his memoir on a Method of Measuring the Capacity of a Condenser*.

The above examples are amply sufficient to exemplify this method of treating problems in networks of conductors, and show how it enables calculations to be made with considerable ease, not only of the distribution of currents and potentials, but of the resistances between any points on a network, the branches of which consist either of simple resistances or of wires having self- and mutual induction with other branches, or of electromagnets, or condensers associated with appropriate commutators.

* This method of Maxwell's, of obtaining the capacity of a condenser has been practically employed, with most excellent results, by Mr. R. T. Glazebrook, F.R.S.; and the full details of the tests to which he subjected the method are given in his paper in the 'Proceedings of the Physical Society,' vol. vi. part iii. p. 204 (June 28, 1884). [Phil. Mag. for August 1884, p. 98.]

XXVIII. *On the Generation of Electric Currents by Sulphur Cells.* By SHELFORD BIDWELL, M.A., LL.B.*

IN a communication recently made to the Physical Society† I ventured the suggestion that the electric conduction of selenium, when prepared in such a form as to be sensitive to light, was, in the literal sense of the term, electrolytic. Selenium itself indeed could hardly be supposed to be an electrolyte; but it was pointed out that when selenium was "annealed" in contact with metallic electrodes, metallic selenides would, at least in most cases, be formed in sufficient quantity to account for the electrolytic phenomena observed; and that even if this were not the case when the electrodes consisted of such a metal as platinum, yet the necessary metallic element might possibly be found in the lead, iron, and arsenic which are contained as impurities in ordinary commercial selenium. Little direct evidence was offered in support of this view; but it was shown that sulphur, when mixed with a certain proportion of sulphide of silver and arranged in the form of a "cell" with silver electrodes, exhibited many of the properties of crystalline selenium, especially that of having its electrical resistance temporarily diminished under the influence of light. Analogy therefore tended to confirm the opinion which I had been led to entertain.

While observing the secondary or polarization-currents which are generated by sulphur cells (as by those made with selenium), after being disconnected from a battery, certain effects were noticed which seemed to indicate that when the electrodes consisted of two different metals, a sulphur cell might be capable of originating and maintaining an independent or primary current. Experiments were therefore made with the object of investigating this point; and the present paper contains an account of the results obtained. I have hardly attempted to connect them together by any complete theory: of some of them, indeed, I can offer no explanation whatever; others appear to be in direct opposition to what might have been expected. But, so far as I have been able

* Read June 27, 1885.

† Phil. Mag. August 1885.

to ascertain, they are entirely novel, and of sufficient interest to be worthy of record.

(1) A slip of mica was wound with two parallel wires of silver and copper 1 millim. apart, and melted sulphur containing a small quantity of precipitated sulphides of silver and copper was spread over one surface. It is not known what proportion of the sulphides was contained in the mixture, because the bulk of them sank to the bottom of the crucible in which the sulphur was melted. When cold, the cell was connected with a reflecting galvanometer, and was found to generate a small but steady current, indicated by a deflection of about 20 scale-divisions. When the connections of the cell were reversed, the current was reversed. It was therefore not due to any thermo-effect in the circuit. The direction of the current was from silver to copper through sulphur: its strength was *diminished* by exposure to light, and *increased* by rise of temperature. Connecting the free ends of the silver and copper wires and heating the junction, it was found that the thermo-current thus produced passed (as usual) from copper to silver through the junction. The increase of current by heat was therefore not to be accounted for by thermo-electric action, for that would produce the opposite effect.

The action is almost certainly of the same nature as that which occurs in an ordinary voltaic cell.

(2) It appeared desirable to construct a cell which, though unsuitable for experiments with light, would have a much smaller resistance than one of the form last described. A plate of copper 3 centim. square was heated, and upon it was spread a mixture consisting of 5 parts of sulphur and 1 part of sulphide of copper. A plate of silver previously heated was then laid on the melted mixture and the two plates squeezed together, thus forming a sandwich-like cell. The thickness of the copper plate was 2.75 millim., of the silver plate .60 millim., and of the completed cell 3.65 millim.; the thickness of the layer of sulphur was therefore .3 millim. When this cell (after cooling) was connected with the galvanometer, the spot of light was violently deflected off the scale. Dr. Fleming was kind enough to make a very accurate measurement of its electromotive force by comparison with one of his standard Daniell cells. It was found to be .0712 volt, and its internal

resistance was 6537 ohms. As in the case of the former cell, the direction of the current is from silver to copper; and there can be no doubt that it is of a voltaic nature. At the time of writing, the cell has been in existence nearly seven weeks, and it is now, I believe, quite as powerful as at first.

(3) It was thought that the internal resistance might be further reduced by adding a larger proportion of sulphide to the sulphur. Another cell was therefore constructed similar in all respects to that last described, except that the sulphur and copper sulphide were mixed in equal proportions. Its internal resistance was enormously lower, being only 13 ohms, but its E.M.F. was also lower, being .0071 volt.

(4) A layer of precipitated sulphide of copper was placed between plates of copper and silver which were squeezed together in a screw-press. The resistance of this arrangement was a small fraction of an ohm; but when connected with the galvanometer, it gave no indication whatever of a current. It seems, therefore, that a certain amount of free sulphur is necessary for the generation of an electromotive force by cells containing copper sulphide.

(5) Two parts of copper sulphide were mixed with one of sublimed sulphur, and the powder was compressed between plates of copper and silver. This cell gave a very small current, indicated by a galvanometer-deflection of 2 or 3 scale-divisions. The deflection was reversed as often as the connections with the binding-screws of the cell were reversed; and the existence of a real, though very small, electromotive force was undoubted. The internal resistance was .088 ohm.

(6) The last-mentioned cell was taken to pieces and remade after the addition of about an equal part of sublimed sulphur to the mixture of sulphur and sulphide. Its resistance was now found to be many megohms, yet it produced a larger galvanometer-deflection than before.

(7) Once more the cell was taken to pieces and a little more sulphide added. When remade, its resistance was at first about 2700 ohms; but it varied considerably. It produced a galvanometer-deflection of about 100 divisions, which in a few minutes increased in a somewhat irregular manner to 250 divisions. It was then disconnected from the galvanometer, and, when again connected after an interval of six

hours, it deflected the spot of light off the scale. Shunting the galvanometer (the resistance of which was 3483 ohms) with a coil of 300 ohms, the deflection amounted to 130 divisions; and on the following day with the same shunt, the deflection was at first about 250 divisions, rapidly diminishing, however, when the circuit was closed.

(8) A cell was made by compressing precipitated silver sulphide unmixed with any free sulphur between plates of silver and copper. When connected with the galvanometer, this cell produced a deflection which, with a shunt of 35 ohms, exceeded 400 divisions. But in this case the *silver* was the negative plate, the direction of the current being from copper through sulphide to silver. The E.M.F. was less than that of the cell described in (2).

(9) Another cell was made in the same manner as that described in (2); but the sulphur was mixed with sulphide of silver instead of sulphide of copper. This gave a strong current in the same direction as that produced when sulphide of copper was used, and opposite to that generated by the cell containing silver sulphide without free sulphur.

(10) Pure sulphur was melted on a clean plate of copper, and, when just liquid, a warmed plate of silver was laid upon it and pressed down with a weight until cold. This cell gave a strong current from silver through sulphur to copper. Sulphides were of course formed during the process of construction.

(11) A melted mixture of sulphur and silver sulphide was spread upon a copper plate, and a plate of silver pressed upon it. When cold, the silver plate was split off with a knife, and a piece of silver-leaf, sufficiently thin to appear blue by transmitted light, was attached (by rubbing) to the exposed surface of the mixture. As before, there was a comparatively strong current from the silver to the copper. The silvered surface was then exposed to the light of burning magnesium wire, and the immediate movement of the spot of light through 50 scale-divisions indicated a *diminution* of the current. When the magnesium was extinguished, the current at once increased to its original strength. When a nearly red-hot brass rod was held at a distance of 3 centims. from the silvered surface, the current slowly *increased* in strength; and when the hot

rod was removed, the current was again slowly diminished. The effects both of light and of heat were verified by many repetitions of the experiments. These results which, so far as regards the effect of light, were unexpected, are of the same character as those described in (1). I have elsewhere* given strong reasons for believing that the combination of sulphur with silver is assisted by the influence of light. If this is so, it is certainly a remarkable fact that increased corrosion of the silver electrode should be accompanied by diminution of the current.

Two days afterwards the silver-leaf had become much discoloured, and was in some parts quite black. When connected with the galvanometer the cell gave a current of nearly the same strength as before; but now it was found to be slightly increased by light as well as by heat; and it is probable that the light as such exerted no influence whatever, the observed effect being really due to the incidental rise of temperature.

(12) The silver-leaf was scraped off, and the surface of the sulphur mixture having been cleaned from all visible traces of free silver with fine emery-cloth, a piece of thin gold-leaf was pressed upon it. It was found very difficult to make it adhere satisfactorily. The cell, when connected with the galvanometer, gave no indication whatever of a current. When the cell described in (2) was also inserted in the circuit, the spot of light was deflected, showing that the first cell was quite able to conduct electricity, and that its failure to originate a current was not owing to bad contact between the gold-leaf and the sulphur mixture.

(13) A cell containing a mixture of sulphur and silver sulphide between plates of silver and iron was found to have an E.M.F. of $\cdot 023$ volt, or about one third of that of the silver-copper cell described in (2). The direction of the current was the same.

(14) A cell was made by melting sublimed sulphur upon a plate of copper and pressing a plate of iron upon the melted sulphur. On connecting it (when cold) with the galvanometer, there was no indication of any current. Nevertheless the cell was found to conduct electricity very well when a

* Phil. Mag. August 1885.

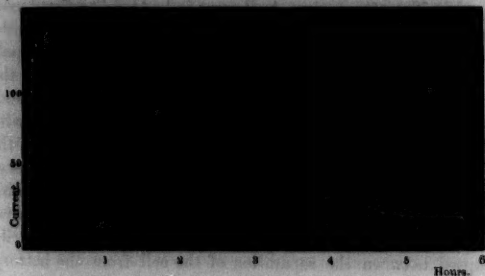
battery was also placed in the circuit. Sulphide of copper was of course formed by the action of the hot sulphur upon the copper.

(15) A melted mixture of 5 parts of sulphur with 1 part of copper sulphide was pressed between plates of silver and iron, and cooled. When first made, this cell generated a sufficiently strong current to deflect the galvanometer-needle as far as the stop would allow; but two hours later, when an attempt was made to measure the E.M.F., it was found to have almost completely disappeared, being less than a hundredth of that of the cell described in (2). The E.M.F. was temporarily restored by connecting the iron and silver plates with the positive and negative terminals of a battery of ten Leclanché cells for a few minutes. The result was of course merely a polarization-current.

(16) Two silver wires A, B were imbedded in a fused mass consisting of equal parts of sulphur and copper sulphide. When cold, the wire A was connected with the carbon pole of a battery of ten Leclanché cells, and the wire B with the zinc pole. After the current had passed for about a second, the cell was detached from the battery and connected with the galvanometer. A current was at once indicated in the direction A B, i. e. in the *same* direction as that of the battery-current which had been caused to pass through the cell. This experiment was repeated many times and on different days, with the same result*. A period of some seconds necessarily elapsed between the separation of the cell from the battery and its attachment to the galvanometer. In order to render this interval as short as possible, the apparatus was so arranged that by depressing a key the transfer could be effected in a small fraction of a second. It was then found that the first effect of the transfer was a strong *momentary* current in the direction B A, which was immediately followed by the more permanent current previously observed in the direction A B. With the view of retarding the first effect, a battery of two Leclanché cells was used, and the connection was made for a longer period. After the battery had been connected for one minute, the cell was transferred to the galvanometer

* The experiments described in the remainder of this paragraph were made after the paper was read.

by means of the key, and the swing of the spot of light through 280 scale-divisions again indicated a current in the direction B A. Though this was not, as before, a current of only momentary duration, it rapidly decreased in strength, becoming zero in almost exactly 30 seconds. After zero was passed, a current was at once set up in the opposite direction A B, which in 30 seconds produced a deflection of 70 divisions, increasing to a maximum of 131 divisions in 10 minutes. The current then slowly diminished; and in $5\frac{1}{2}$ hours after the commencement of the experiment the deflection had fallen to 17 scale-divisions. At this point the observations were discontinued. In the accompanying curve the abscissæ represent the time in hours, and the ordinates the current in scale-divisions. The current of 30 seconds' duration in the direction B A is not represented.



On another occasion, when the two Leclanché cells had been connected for 3 minutes, the secondary current from B to A decreased still more slowly than in the last-mentioned experiment, vanishing in $3\frac{1}{2}$ minutes. The current which followed in the direction A B attained its maximum, indicated by 155 divisions, in $6\frac{1}{2}$ minutes, and then steadily decreased for nearly 4 hours, when it again became zero. The spot of light did not, however, remain stationary, but moved steadily on, indicating a second reversal of the current. In one hour after zero had thus been crossed for the second time, the galvanometer-deflection was 12 scale-divisions; and $4\frac{1}{2}$ hours later, when the last observation was made, it had increased to 20 divisions. In both these experiments the galvanometer

(of which the resistance was nearly 3500 ohms) was shunted with a coil of 100 ohms.

These alternating currents are probably of the same nature as those which Faraday found to be generated when copper and silver, or two pieces of copper, or two pieces of silver, were immersed in a solution of potassium sulphide*. The wires are alternately protected from the action of the free sulphur by an investing coat of sulphide.

Summary.

Plates of silver and copper imbedded in a mixture of sulphur with sulphides of copper or silver constitute a cell which at the ordinary temperature is capable of generating and maintaining a constant current, the silver being the positive element. Such a cell, in which the mixture consisted of 5 parts of sulphur with 1 of copper sulphide between plates 3 centim. square and .3 millim. apart, had an E.M.F. of .0712 volt and an internal resistance of 6537 ohms.

If the proportion of copper sulphide to sulphur is increased, the internal resistance of the cell is diminished; but its E.M.F. is also diminished.

A cell containing copper sulphide unmixed with free sulphur fails to produce any appreciable current.

A cell containing silver sulphide only generates a current in the opposite direction to that produced when the sulphide is mixed with free sulphur.

Copper used in conjunction with iron or gold gives no current whatever at the ordinary temperature.

The current generated by a silver-copper cell containing free sulphur mixed with sulphide is diminished by the action of light and increased by heat. It has not been ascertained whether the effect is upon the E.M.F. or the internal resistance, or both.

If a battery-current is caused to pass for a short time through a cell consisting of two silver electrodes imbedded in a mixture of sulphur and copper sulphide, the cell, after being disconnected from the battery, will generate a current of very short duration in the direction opposite to that of the battery-current, followed by a current which may be maintained for several

* Exp. Res. §§ 1911 and 2036.

hours in the *same* direction as the battery-current. A second reversal has in one case been found to occur after an interval of four hours.

The experiments, of which an account is here given, must be considered as being merely of a preliminary nature. A complete investigation of the phenomena in question would require more time than I am at present able to devote to the subject.

Addition, August 3rd.

From (4), (5), and (6) it appears that a cell consisting of copper and silver plates is incapable of generating a current unless a certain proportion of free sulphur is mixed with the sulphide. Thinking that the function of the free sulphur might be merely to form silver sulphide by direct combination with the silver, I constructed a cell as follows:—A layer of copper sulphide was spread upon a plate of copper; a polished steel plate was laid upon the sulphide, and the whole was strongly compressed in a vice. The steel plate was then removed, and a thin layer of silver sulphide was spread upon the smooth surface of the copper sulphide. The cell was completed by pressing a silver plate upon the silver sulphide. This was found upon trial to give a current which, with an external circuit of low resistance, was many times stronger than that generated by any of the cells previously made. It seems to be exactly analogous in its action to a Daniell cell consisting of plates of zinc and copper in solutions of zinc sulphate and copper sulphate. The quantity of the copper sulphide would be gradually diminished, copper being deposited on the copper plate, while the quantity of silver sulphide would continually increase with consumption of the silver.

In conclusion, it seems probable that, by selecting such metals as experiment might prove to be better suited for the purpose than silver and copper, a cell might be constructed upon the principle of that described in the above paragraph which would be of practical and commercial value.

XXIX. *On the Law of the Electromagnet and the Law of the Dynamo.* By SILVANUS P. THOMPSON, D.Sc., B.A.*

It is remarkable, considering the vast amount of research that has been made, that authorities in physics are in so little agreement as to what the law of the electromagnet is. Upon the one topic that is all-essential to the electric engineer the professed physicist is either silent, or gives mathematical rules that are hopelessly wide of the observed facts, and based in most cases on no consistent theory. The expressions given by Weber† for the relation between the magnetizing force and the induced magnetism are an exception; but they are unmanageable in the extreme, and have led to no useful results. The expression given by Lamont‡ is also based on a certain consistent theory, but it appears to be entirely unknown both to physicists and to electric engineers, and is to be found only buried in the depths of Lamont's own treatise. The two mathematical formulæ which are most often given in treatises on physics as representing the relation between the magnetizing current and the induced magnetism of the electromagnet are the following:—

Lenz and Jacobi's Formula.—According to the experiments of these early investigators (1839), the magnetism of the electromagnet is simply proportional to the strength of the current and to the number of turns of wire in the coil. This may be written as

$$m = kSi,$$

where i is the strength of the current, S the number of turns in the coil, k a constant depending on shape and quality of iron, and m the strength of the pole. Joule (1839) showed this rule to be incorrect, and that as the iron became saturated, m ceased to be proportional to Si .

Müller's Formula.—Müller gave an expression of the form:—

$$m = b \tan^{-1} \frac{Si}{a},$$

* Read November 14, 1885.

† Weber, *Elektrodynamische Maasbestimmungen*, p. 572. See also Maxwell, 'Electricity and Magnetism' (2nd edition), vol. ii. p. 78.

‡ Lamont, *Magnetismus*, p. 41.

where b and a are constants, depending on the form, quantity, and quality of the core and on the form and disposition of the coils. Very similar formulæ have been used by Von Waltenhofen, Dub, Cazin, and Breguet. They are purely empirical, fail to represent the facts with accuracy, and have moreover the defect of not lending themselves conveniently to use in the equations of electromagnetic motors, dynamo-electric machines, and other kinds of electric machinery.

More recently another formula has been coming into use, and is known—for reasons which will presently appear—as Frölich's formula: it is written

$$m = \frac{i}{a + bi},$$

where a and b are again constants, b being the reciprocal of the maximum value of m . This formula, like the preceding, is a perfectly empirical expression not based in itself upon any physical theory, and it accords with the observed facts of the electromagnet better than any of the preceding formulæ. Frölich, who in 1878 was using merely Lenz and Jacobi's formula*, appears to have adopted this expression in 1880†, in consequence of the experiments which he had made with dynamo-machines. A similar formula had been used twenty-five years before by Robinson‡ to express the lifting-power of an electromagnet, and similar formulæ have been used by Oberbeck§, Fromme||, Clausius¶, Ayrton and Perry**, and Rücker††.

Now it must be obvious that since almost the whole of electric technology deals with the electromagnet and its applications, it is of utmost importance to know what the true law of the electromagnet is. If the technologist is left to the mercy of the physicist, he has to choose between formulæ

* Frölich, *Die Lehre von der Elektrizität und dem Magnetismus*, p. 237.

† Frölich, *Berl. Berichte*, Nov. 18, 1880, p. 973; also *Elektrotechnische Zeitschrift*, 1881, April, p. 139.

‡ Robinson, *Trans. Roy. Irish Academy*, vol. xxii. p. 1, 1855.

§ Oberbeck, *Pogg. Ann.* cxxxv. pp. 74-98, 1868.

|| Fromme, *Pogg. Ann.* clv. p. 305, 1878.

¶ Clausius, *Wied. Ann.* xx. p. 367, 1883.

** Ayrton and Perry, *Journ. Soc. Electr. Eng. and Electr.* xii. p. 317, 1883.

†† Rücker, *Phil. Mag.* ser. 5, xix. p. 463 (June 1885).

that are based on physical conceptions and which are utterly unmanageable for his purpose, and formulæ that are purely empirical expressions and have no rational basis.

But the study of the modern dynamo-electric machine has brought us to a point that makes some decision a necessity. The entire action of the dynamo is dependent on the magnetizing action of the electric current, and the law of the dynamo is, in the very nature of things, not to be discovered, save by the discovery of the law of the electromagnet. It is indeed extraordinary that such able physicists as Mascart and Angot, Mayer and Auerbach, Schwendler, and Herwig sought in vain for the true law to connect the electromotive force of the dynamo with its speed, the resistance of its circuit, and the constants of its construction. Hopkinson came nearest to the mark when in 1879 he described the properties of the dynamo in terms of those of a certain curve, which we now call the characteristic curve of the machine; but he did not state the law of the dynamo algebraically, and apparently he accepted as true Weber's formula for the electromagnet.

But the law of the dynamo is now known, and expressions have been deduced which are found to agree with the utmost accuracy with the observed facts. The discovery is almost entirely due to Dr. Frölich*, whose papers, published at intervals from 1880 to 1885, are in their way already classical. The results first obtained by Frölich related exclusively to that form of dynamo in which the armature-coils and the field-magnet coils are both included in the main circuit—the series-dynamo—though they have been further extended by the present writer, and by others, including Frölich himself, to other forms of machine. A very brief résumé of Frölich's research will explain his method of arriving at the law.

His theory is based upon (1) Faraday's law of induction, (2) Ohm's law, (3) a curve, called by him the current-curve, expressing certain results of experiments made on the series-wound dynamo.

Following Faraday's principle, the induced electromotive

* Frölich's chief papers are as follows:—*Berl. Berichte*, 1880, p. 978; *Elektrotechnische Zeitschrift*, vol. ii. p. 134 (1881); *ib.* vol. iii. pp. 69 and 113 (1882); *ib.* vol. iv. pp. 60, 67, and 71 (1883); *ib.* vol. vi. pp. 128, 130, 227 (1885).

force E will be proportional to the speed of the machine n and to a quantity M which Frölich calls the "effective magnetism," and which is itself proportional to the effective area of the armature-coils and to the intensity of the magnetic field. Writing this as an equation, we have

$$E = nM,$$

and, by Ohm's law,

$$E = iR,$$

which gives us for the current i ,

$$i = \frac{n}{R} M,$$

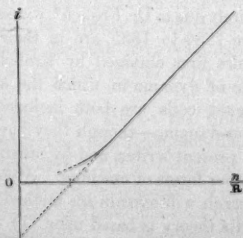
or

$$\frac{i}{M} = \frac{n}{R}.$$

Since M is itself a function of i , and not primarily of n or R , it is clear that i is itself a function of n/R , and we may write

$$\phi(i) = \frac{n}{R}.$$

This function Frölich set himself to investigate by pure experiment without making any hypothesis whatever. He determined the values of i at various speeds and with various resistances, and plotted out the results as a curve, values of i being taken as ordinates and values of n/R as abscissæ. This curve (see figure) he termed the "current-curve." It shows itself to be very nearly a straight line, which, however, does not pass through the origin.



The portion which departs from the line is difficult of observation because it is only obtained when either n is very small or R very large, that is to say when the dynamo is scarcely able to excite its magnets. Neglecting this unstable state, and dealing only with that part of the curve which relates to the machine as it is when in action, it is clear that the relation between the two variables may be expressed as

$$bi = \frac{n}{R} - a,$$

where b and a are constants ; giving us as the equation of the series-dynamo :—

$$i = \frac{1}{b} \left(\frac{n}{R} - a \right).$$

Substituting for n/R its equivalent i/M , Frölich then obtained from the law of the dynamo the following expression for the “effective magnetism” :—

$$M = \frac{i}{a + bi},$$

an expression which he himself appears, in his original paper, to have considered as a mere interpolation-formula. It is, however, this expression which is generally referred to as Frölich's formula for the electromagnet.

It was in 1883 that the author of this paper, in working at the problem of the dynamo, realized the significance of this result of Frölich's. He had been using as the fundamental law

$$E = 4\pi AH$$

to express the average value of the electromotive force developed by rotating in a uniform magnetic field of intensity H , an armature whose total effective area was A ; and requiring a formula to connect H with the exciting current, he was considering the various equations of Müller, Weber, and others when he became aware that Frölich's expression furnished what was needed ; and, finding it necessary for the case of compound-wound machines to express the excitation in terms of ampère-turns rather than of ampères, he rewrote the formula in the following way :—

$$H = GSi \frac{k}{1 + \sigma Si^2}$$

where, as before, Si is the number of ampère-turns, G a geometrical coefficient, k the coefficient of magnetic permeability in the initial stage, and σ a small saturation-coefficient dependent on the quantity and form as well as on the quality of the iron of the core.

Writing similarly Zi_s for the ampère-turns of a shunt-winding, the expressions for H became

$$H = GZi_s \frac{k}{1 + \sigma Zi_s}$$

for shunt-wound machines, and

$$H = G(Si \pm Zi_s) \frac{k}{1 + \sigma(Si \pm Zi_s)}$$

for compound-wound machines.

Inserting these values in the fundamental equation at once gave the following expressions:—

Series dynamo.

$$E = \frac{1}{\sigma} \left\{ 4nAGk - \frac{\Sigma R}{S} \right\}$$

$$i = \frac{1}{\sigma} \left\{ \frac{4nAGk}{\Sigma R} - \frac{1}{S} \right\}$$

Shunt dynamo.

$$E = \frac{1}{\sigma} \left\{ 4nAGk - \frac{r_a r_s + Rr_s + Rr_a}{ZR} \right\}$$

Compound dynamo (at critical speed for regulation).

$$e = \frac{1}{\sigma} \left\{ \frac{r_s(r_a + r_m)}{S(r_s + r_a)} - \frac{r_s}{Z} \right\} \text{ (short shunt),}$$

or

$$e = \frac{1}{\sigma} \left\{ \frac{r_s(r_a + r_m)}{S(r_s + r_a + r_m)} - \frac{r_s}{Z + S} \right\} \text{ (long shunt),}$$

where r_a , r_m , r_s , and R are the resistances of armature, main-circuit coil, shunt-coil, and external circuit respectively, and e the difference of potentials at the terminals of the machine. These results were announced by the author at the Montréal meeting of the British Association, and were published by him in his volume on *Dynamo-Electric Machinery*, as was also the important proposition that the factor $\frac{1}{\sigma}$ (which may be called *the determinant* of the working power of the dynamo, since it appears in every one of the equations, both of current and electromotive force) is that number of ampère-turns which will reduce the effective permeability of the

electromagnet, or rather of the magnetic circuit, to half its initial value. It will be seen that all these synthetical results were new, except that for the current of the series dynamo, which is identical with the expression that had been deduced by Dr. Frölich from the current-curve which represented his experimental results.

Since the publication of the author's treatise, Frölich has further elaborated the equations of the compound-wound machines; and a more generalized form has been deduced by Prof. Rücker in the masterly paper which he communicated to the Physical Society in March last, in which, however, he attributes to Frölich (I know not on what grounds) the general equation for the electromotive force of all kinds of dynamos. Frölich has also verified the adequacy of these formulæ by comparing results calculated from them with actual experiments on compound-wound machines. One example will suffice.

<i>n.</i>	<i>R.</i>	<i>e</i> obs.	<i>e</i> calc.	<i>i</i> obs.	<i>i</i> calc.
850	0.841	127	127	151.0	151.0
853	1.22	133	133	109.0	109.0
855	2.34	139	140	59.9	59.4
850	4.52	138	139	30.3	30.5
850	9.25	136	138	14.7	14.7
850	22.1	137	136	6.2	6.2
845	222.0	133	134	0.6	0.6

Such a series of results proves the law of the dynamo to be fully established. But if the law of the dynamo, so deduced from Frölich's empirical expression for the electromagnet, be true, *then*, to an equal degree of precision, *must the law of the electromagnet be itself true.*

But Frölich's expression is not the expression of any physical law: it is a mere interpolation-formula, destitute of rational significance. What then is the true law of induced magnetism? And how comes it that an expression devoid of physical significance so nearly expresses the physical law?

That question the author believes to have been already

solved by a forgotten investigation of Lamont, published almost without note or comment in his *Handbuch von dem Magnetismus* in 1867. On p. 41 of that work he gives the following equation:—

$$m = \frac{aMx}{M + ax},$$

as a convenient first approximation to another equation based on a new theory of induced magnetism. In the above equation x stands for the magnetizing force, m the induced magnetism, and M the maximum value of the latter. Writing Gk for a , Si for x , and σ for $\frac{a}{M}$, we at once get

$$m = \frac{GkSi}{1 + \sigma Si},$$

which is the author's way of writing Frölich's equation.

The physical theory which led Lamont up to this result is of great interest, the more so since it has been revived during the current year in a more elaborate form by Bosanquet*, who has apparently been led thereto by an independent line of thought, and who has gone much further than Lamont in bringing in the additional consideration of the forces resisting the orientation of the molecules. Lamont's theory is, briefly, that the permeability of the iron diminishes as the permeation increases, being at every stage of the magnetization proportional to the deficit of saturation. He assumes that for every bar there is a certain maximum of magnetization, to which it could only attain under the influence of an infinitely great magnetizing force; and that the permeability of the bar is at every stage of the magnetization proportional to the difference between the actual magnetization and the possible magnetization. This is, in other words, as if in every bar there were room for only a certain limited number of magnetic lines of force, and that, when any lesser number have been induced in it, the susceptibility of the bar to the reception of additional lines is proportional to the room yet left for them in the bar. Lamont's theory is expressed as follows: Let the magnetism present at any stage be called m , and let the maximum magnetism be called M . Then the

* Bosanquet, *Phil. Mag.* ser. 5, xix. p. 85 (Feb. 1885).

amount which the bar can still take up is $M-m$; and it is to this that the permeability dm/dx is proportional. Here x may be understood, as itself proportional to the number of ampère-turns of the exciting current; and we may write

$$\frac{dm}{dx} = k(M-m),$$

where k is a constant depending on the units employed, and on the initial value of the permeability of the magnetic circuit when $m=0$. Integrating we obtain

$$M-m = A\epsilon^{-kx},$$

where A is a constant of integration. But when $x=0$, $m=0$ also; whence $A=M$, giving

$$m = M(1 - \epsilon^{-kx}),$$

which is Lamont's formula.

Lamont also develops this expression in ascending powers of kx ,

$$m = M k x \left(1 - \frac{kx}{1.2} + \frac{k^2 x^2}{1.2.3} - \dots \right),$$

and he then makes the remark that it may be approximately represented by the simpler, but empirical expression

$$m = \frac{a M x}{M + a x},$$

which is identical in form with the expression known as Frölich's. Now, writing k for a/M ,

$$m = \frac{M k x}{1 + k x},$$

and expanding this in ascending powers, we get

$$m = M k x (1 - kx + k^2 x^2 - \dots).$$

Neglecting the fourth and higher terms, it will be seen that the formulæ are very nearly equal for all small values of kx , and are identical for the value

$$kx = \frac{3}{5},$$

or when the actual magnetism has about 456 of the value it would have under an infinite magnetizing force. For larger values of the magnetizing force the values of m calculated by

the empirical formula are slightly greater than those calculated by the exponential formula.

None of these formulæ render any account of a phenomenon noticed in many cases where an electromagnet is magnetized with gradually ascending magnetic forces, namely, a concavity in the *early* part of the curve of magnetization, the permeability apparently becoming greater after a certain degree of magnetization has been attained. The researches of Chwolson and of Siemens seem to show that this apparent increase in the permeability (which is not observed with descending magnetic forces) is due to non-homogeneity, and to the resistance of some of the molecules to magnetization. Bosanquet's theory may be taken in connection with the recent observations of Rowland, Warburg, Ewing, and Hopkinson on this subject, which, however, is of no great importance here since the apparent maximum of permeability is attained in most cases at a much lower degree of magnetization than that at which the dynamo is actually worked.

The very close agreement between the observed values of current and potential of the dynamo and those calculated by the equations based upon Frölich's formula, proves that nowhere within the working range do the actual values of the magnetization differ sensibly from those calculated by the Frölich formula. That is to say this formula may be taken as *something more than a first approximation* to the true law of the electromagnet for all degrees of saturation of the magnets employed in practice, and may be taken as a first approximation for all degrees of saturation beyond that usually attained.

It is eminently desirable, in the face of these facts, that the imperfect formula of Lenz and Jacobi, as well as that of Müller, should henceforth disappear from the text-books of physics, and that instead thereof the true law of the electromagnet should be stated, either in the exponential form or in the simpler form that is found to be equally true within the range of saturation attained in practice.

XXX. *A Note on the Electromotive Force of certain Tin Cells.*

*By E. F. HERROUN, of King's College, London.**

HAVING failed to find any published account giving the electromotive force of cells consisting of tin in solutions of any of its salts, opposed to other metals in solutions of their corresponding salts, I thought that some measurements recently made by myself might be of interest to this Society as tending to complete the series of double-fluid cells already investigated and recorded by others.

The cells examined consisted of an outer containing-vessel of glass or glazed earthenware, in which was placed one of the metals in a solution of one of its salts, and an inner porous pot, 8 centim. high, containing the other metal in a solution of its corresponding salt. It will be seen that they were, therefore, similar in construction to the ordinary laboratory form of Daniell cell, and the standard employed in these experiments was a Daniell cell made up in exactly the same manner.

The solutions in the Daniell cell were copper and zinc sulphates of equal molecular strength as recommended by Dr. C. Alder Wright, the proportion being $1.8 \text{ M}''\text{SO}_4$, $100 \text{ H}_2\text{O}$. The cell made up in this way was assumed to have an electromotive force of 1.08 volt, and it probably differed but little from this value.

Method of Measurement.

The measurements of the E.M.F. of the cells were made by the zero method of Latimer Clark, in which no polarization is possible when the adjustment is effected. The standard Daniell cell was used before and after each experimental cell in order to avoid error due to variation in the potential of the wire, resulting from changes in the E.M.F. or internal resistance of the third or polarizing battery, which consisted either of two Daniell cells or a Fuller's bichromate. This precaution, however, was usually found to be superfluous, as the circuit of the third battery was only closed when an observation was

* Read November 14, 1885.

being made, and its E.M.F. and internal resistance varied only within very small limits.

In order to ascertain if the E.M.F.'s measured were really due to the essential chemical changes occurring within the cell, it was usual, after making an observation with an experimental cell when first put up, to allow the cell to work for a short time on a circuit of a few ohms, so as to cause the surfaces of the metals to become dissolved and deposited upon respectively. The circuit was then opened, and after a short rest the E.M.F. was again determined. If this differed by little or nothing from the previous measurements it was considered to be the true value for that cell; if, on the contrary, there was an appreciable difference further measurements were made. By operating in this way it was found that copper plates (particularly electro-deposited metal) which had been exposed to the air, and had acquired a superficial film of suboxide, usually gave, when first put up, an E.M.F. somewhat in excess of their true value. After allowing the cell to send a current for a few minutes, the E.M.F. fell to the normal value, and thereafter remained sensibly constant. (*Vide* also Dr. Fleming, "Use of Daniell's Cell," *Proc. Phys. Soc.* vol. vii. p. 168.)

As such initial fictitious electromotive forces would affect to their full extent the results obtained by a method like that of Poggendorf or Latimer Clark, or in fact any method in which difference of potential and not the current is measured, it was thought well to control such measurements with one or two cells by using Wheatstone's method of adding a resistance to bring down the deflection of a galvanometer-needle through a given number of degrees. The results obtained by this means agreed remarkably with those found by the previously mentioned process.

In the calculations relating to the heat of formation of some of the salts, it has been taken as a mean value that the electrochemical equivalent of hydrogen is $\cdot 0001038^*$, and that the Joule equivalent is $4\cdot 16 \times 10^7$ ergs. From these numbers the factor which, multiplied into the heat, expressed in gramme-degrees, evolved by the displacement of the

* "On the Determination of Chemical Affinity in terms of Electromotive Force," by Messrs. Wright and Thompson, *Phil. Mag.* vol. xix. p. 7.

gramme equivalent of one metal by that of another, will give the electromotive force in volts, is found to be 4.318×10^{-5} or 4.32×10^{-5} nearly.

The electromotive forces of the following combinations have been measured :—tin and copper, zinc and tin, cadmium and tin, in solutions of their sulphates and chlorides ; and zinc and tin in solutions of their iodides.

The stannous sulphate employed was obtained by the action of sulphuric acid with the aid of heat upon finely granulated tin, in presence of platinum or copper to aid the action. The solution of stannous sulphate contained some free sulphuric acid ; and in fact it appears to be impossible to prepare a neutral solution suitable for these experiments, as on dilution to the required strength it deposits a basic salt, the solution becoming acid. Moreover a nearly neutral solution of stannous sulphate when kept is easily acted on by atmospheric oxygen, becoming cloudy and finally depositing what is probably either a mixture of basic stannous sulphate and stannic hydrate, or a basic stannic sulphate (?). With free sulphuric acid present in the proportion of from 1 to 2 per cent., the solution was satisfactory, and the effect of the free acid is considered below. For similar reasons it was found necessary to operate with solutions of stannous chloride containing a small amount of free hydrochloric acid, and with stannous iodide containing free hydriodic acid.

The metallic tin employed as plates in these cells was chemically pure, it having been obtained by electrolysis of a pure solution of the sulphate. The electro-deposited metal was carefully fused and rolled into strips.

The copper plates employed had in all cases been well covered with the electro-deposited metal. The zinc used was a good specimen of commercially "pure" metal, and was amalgamated. The cadmium was fairly pure, and was covered with the electro-deposited metal.

Electromotive Force of Cells consisting of the Metals immersed in Solutions of their Sulphates.

I. Zinc-Tin Sulphate Cell.

A cell in which zinc in $.5\text{ZnSO}_4$, $100\text{H}_2\text{O}$ is opposed to tin

in $\cdot 5\text{SnSO}_4$, $100\text{H}_2\text{O}$, both solutions containing about 1·5 per cent. H_2SO_4 , was found to have an E.M.F. of from $\cdot 519$ to $\cdot 535$ volt, the variation being due to two causes: (1) slight differences in the percentage of free H_2SO_4 , and (2) differences in the condition of the surface of the tin plate.

It is thus difficult to assign an exact value except between such limits; but with solutions of the above strength, and with a tin plate covered with freshly deposited tin, the value of $\cdot 525$ volt may be taken as a tolerably accurate average E.M.F. for this kind of cell.

It would, of course, be unphilosophical to assume that an *equality* in the amount of free acid present in each solution has the same effect as operating with neutral solutions; but being unable to employ a neutral solution of stannous sulphate, the addition of an amount of sulphuric acid to the zinc solution equal to that present in the SnSO_4 solution was considered to be the fairest way of meeting the difficulty.

Increasing the amount of free sulphuric acid in the tin solution only has naturally the effect of lowering the E.M.F., so that with 5 per cent. H_2SO_4 the E.M.F. is not higher than $\cdot 49$ volt.

Substituting a 10 per cent. solution of H_2SO_4 for the zinc sulphate, leaving the stannous sulphate unchanged, raises the E.M.F. to $\cdot 596$ volt.

The effect of varying the strength of both solutions (keeping them of equal molecular strength) is the same in kind as that which occurs with the zinc-tin chloride cell, but is less marked. (See Zinc-Tin Chloride Cell.)

The heat of formation of stannous sulphate being unknown, it is impossible to compare the value $\cdot 525$ volt with deductions from thermal data.

II. Tin-Copper Sulphate Cell.

A cell in which tin in $\cdot 5\text{SnSO}_4$, $100\text{H}_2\text{O}$ is opposed to copper in $\cdot 5\text{CuSO}_4$, $100\text{H}_2\text{O}$, containing from 1 to 2 per cent. H_2SO_4 in both solutions, was found to have an E.M.F. varying from $\cdot 56$ to $\cdot 572$ volt, according to the condition of the surfaces of the plates and the acidity of the solutions. The effect of increasing the proportion of free H_2SO_4 in the tin solution is to raise the E.M.F. of the cell, and it is, of

course, opposite to the effect observed in a cell in which zinc displaces tin. The increase of free sulphuric acid equally in both tin and copper solutions also tends (at least through a certain range up to 3 or 4 per cent.) to slightly raise the E.M.F.; and from this I conclude that with neutral solutions of the above strength, the value would not exceed, and would probably fall slightly below, $\cdot 56$ volt. Preference would therefore be given to the minimum value $\cdot 56$ instead of the mean value $\cdot 566$.

Alteration of the strength of the solutions is complicated by the question whether such alteration involves a change in the percentage of contained acid or only of the salts employed. Increase of strength of the solutions as regards the salts only, is attended with a slight depreciation of the E.M.F.; but if the addition of more salt causes an increase in the acidity of the tin solution, the effect is to heighten the E.M.F., so that these opposite effects may equal each other, or one may exceed the other, according to their relative magnitude.

Cells were set up in which tin in a solution of 20 per cent. H_2SO_4 (sp. gr. 1.141) was opposed respectively to tin in $\cdot 5\text{SnSO}_4$, 100 H_2O , containing $1\frac{1}{2}$ per cent. H_2SO_4 , and to $\cdot 5\text{CuSO}_4$, 100 H_2O , and the following values obtained:—

Tin-tin sulphate cell. $\cdot 058$ – $\cdot 062$ volt.

Tin sulphuric-acid-copper sulphate cell. $\cdot 59$ volt.

(It will be seen that the last-mentioned cell does not exceed the tin-copper sulphate cell ($\cdot 566$) by an amount equal to the tin-tin cell, the lower E.M.F. being probably due to an adverse E.M.F. resulting from diffusion into the comparatively strong sulphuric-acid solution.)

If the mean values assigned to the zinc-tin cell and the tin-copper cell are added together, they are found to exceed the zinc-copper cell, which they should theoretically equal

$$(\cdot 525 + \cdot 566 = 1\cdot 091 - 1\cdot 08 = \cdot 011 \text{ excess}).$$

This result confirms the opinion that for neutral solutions the value $\cdot 566$ is probably too high for the tin-copper cell, and justifies the minimum value $\cdot 56$ being taken as correct. This still gives a slight excess $1\cdot 085 - 1\cdot 08 = 0\cdot 005$, and in practice it is found that a zinc-tin cell and a tin-copper cell

united in series actually give an E.M.F. slightly in excess of the zinc-copper cell. This difference may be due either to the different conditions of the surfaces of the two tin plates or to some less obvious cause; but as it has been found to exist on several occasions in different cells, it can hardly be a mere coincidence.

III. Cadmium-Tin Sulphate Cell.

A cell set up with cadmium in $\cdot 5\text{CdSO}_4$, $100\text{H}_2\text{O}$, opposed to tin in $\cdot 5\text{SnSO}_4$, $100\text{H}_2\text{O}$, both containing about 1.5 per cent. H_2SO_4 , was found to have an E.M.F. of $\cdot 189$ volt.

In order to compare the summation with the zinc-tin cell, a determination of the zinc-cadmium sulphate cell was made (using $\cdot 5\text{M}''\text{SO}_4$, $100\text{H}_2\text{O}$), and it was found to have an E.M.F. of $\cdot 335$ volt. (The value given by Messrs. Wright and Thompson for the zinc-cadmium cell where $m = \cdot 5$ and the solutions are *neutral* is $\cdot 361$, which, correcting for difference of standard, gives $\frac{1\cdot 08}{1\cdot 114} \times \cdot 361 = \cdot 349$ or somewhat higher.)

Adding together the values obtained we get $\cdot 189 + \cdot 335 = \cdot 524$ volt for the zinc-tin cell, a result which shows a striking agreement with the average value previously given, viz. $\cdot 525$.

The average E.M.F. of the cadmium-copper cell was also determined and found to be $\cdot 744$, Messrs. Wright and Thompson finding $\cdot 753$ for the same cell, which gives $\left(\frac{\cdot 753 \times 1\cdot 08}{1\cdot 114} = \right)$ $\cdot 73$ volt; or, in other words, my value for the cadmium-copper is as much above, as my zinc-cadmium is below, their estimate, both values adding up correctly,

$$\cdot 73 + \cdot 349 = 1\cdot 079,$$

and

$$\cdot 744 + \cdot 335 = 1\cdot 079.$$

This discrepancy is no doubt due to the fact that the solutions employed by me contained about 1.5 per cent. of free H_2SO_4 in order to keep the conditions uniform with the tin cells.

Subtracting the E.M.F. of cadmium-tin from that of cadmium-copper, we get the E.M.F. of tin-copper, thus:—

$$\cdot 744 - \cdot 189 = \cdot 555 \text{ volt};$$

which accords fairly well with .56, but is, as would be anticipated, somewhat less.

Cells the algebraic sum of whose E.M.F.'s should equal 1.08, the standard Daniell.

$$\begin{array}{rcl} \text{Zinc-tin-sulphate} & + & \text{tin-copper-sulphate.} \\ .525 & + & .56 \end{array} = 1.085.$$

$$\begin{array}{rcl} \text{Zinc-cadmium} & + & \text{cadmium-tin} & + & \text{tin-copper.} \\ .335 & + & .189 & + & .56 \end{array} = 1.084.$$

$$\begin{array}{rcl} \text{Zinc-tin} & + & \text{cadmium-copper} & - & \text{cadmium-tin.} \\ .525 & + & .744 & - & .189 \end{array} = 1.08.$$

Electromotive Force of Cells consisting of the Metals in Solutions of their Chlorides of Equal Molecular Strength.

I. Zinc-Tin Chloride Cell.

A cell set up with amalgamated zinc opposed to tin in solutions of their chlorides of the strength .5 M¹⁰⁰Cl₂, 100 H₂O, containing 1 per cent. HCl, gave an E.M.F. of .544 to .553 volt, mean = .549 volt.

Diluting the solutions equally with distilled water causes a rise of E.M.F. up to about .607 volt, but dilution with 1 per cent. hydrochloric acid causes a slight diminution of E.M.F. This and other facts render it probable that with this cell, and to a less extent with the zinc-tin sulphate cell, increase of solution-strength without increase in the percentage of contained acid is associated with a rise of E.M.F.; but that with a corresponding increase of acidity, with a fall of E.M.F. Thus it is that dilution with distilled water raises the E.M.F. not because the metallic salt is diluted, but because the free acid is more attenuated.

Mutatis mutandis the same remarks apply to tin-copper or other cells in which tin is the metal attacked (*i. e.* with constant acidity and increasing strength of solution the E.M.F. falls, but with increase of *both* the E.M.F. rises, or does not fall).

The heat of formation of zinc chloride and of stannous chloride being known, the theoretical E.M.F. is directly calculable.

$$[\text{Zn, Cl}_2, \text{aq.}] = 112840. \quad [\text{Sn, Cl}_2, \text{aq.}] = 81140.$$

$112840 - 81140 = 31700$; half of which multiplied into 4.32×10^{-5} gives $.684$ volt as the E.M.F. of zinc displacing tin from solution of its chloride, and the difference between the observed and calculated value is $.684 - .549 = .135$ volt. Or to employ the terminology of Messrs. Wright and Thompson, $-.135$ is the "*Thermovoltaic constant*" for tin in solution of its chloride for certain strengths of solution (near $.5 \text{ MCl}_2$, $100 \text{ H}_2\text{O}$).

II. Cadmium-Tin Chloride Cell.

A cell set up with electro-cadmium opposed to tin in solutions of their chlorides, $.5 \text{ M}''\text{Cl}_2$, $100 \text{ H}_2\text{O}$, was found to give an E.M.F. of $.247$ to $.262$ volt and an average value of $.249$ volt. The solutions used contained the same amount of free hydrochloric acid, namely about 1 per cent.

The heats of formation being $[\text{Cd}, \text{Cl}_2, \text{aq.}] = 96250$ and $[\text{Sn}, \text{Cl}_2, \text{aq.}] = 81140$ and their difference 15110 or 7555 per equivalent, the theoretical E.M.F. of cadmium displacing tin from dilute solutions of its chloride is $.326$ volt, and the difference between this and the average observed value is $.077$.

The E.M.F. of a zinc-cadmium chloride cell was also determined and found to be between $.291$ and $.308$ volt, giving an average of $.301$ volt. This is below the value given by Messrs. Wright and Thompson, owing to the solutions used by me containing 1 per cent. free HCl , they giving for this cell $.329$ volt, which becomes $\left(\frac{.329 \times 1.08}{1.114} = \right) .318$ volt, allowing for difference of standard.

In reference to a point to which these authors* drew attention a short time ago, namely, that, although the heat evolved by displacing cadmium by zinc in solutions $.25 \text{ CdCl}_2$, $100 \text{ H}_2\text{O}$ is greater than that evolved by the same substitution in the case of the sulphates, yet the E.M.F. of the zinc-cadmium *chloride* cell is materially less than that of the zinc-cadmium *sulphate* cell, my measurements afford direct confirmation, as the value given here for the average E.M.F. of the zinc-cadmium sulphate cell is $.335$ volt and that for the zinc-cadmium chloride cell only $.301$ volt, or a difference of $.034$ volt, substantially the same as the difference given by them, viz. $.035$.

* Phil. Mag. vol. xvii. p. 378 (May 1884).

III. *Electromotive Force of Zinc-Tin Iodide Cell.*

A cell set up with amalgamated zinc in $\cdot 25 \text{ ZnI}_2$, $100 \text{ H}_2\text{O}$ opposed to tin in $\cdot 25 \text{ SnI}_2$, $100 \text{ H}_2\text{O}$, both solutions acidified with 1·5 per cent. of HI, was found to have an E.M.F. of from $\cdot 47$ to $\cdot 49$, averaging $\cdot 485$ volt.

There being no thermal data at my disposal no comparisons can be drawn.

XXXI. *Note on the Verification of Thermometers at the Freezing-Point of Mercury.* By G. W. WHIPPLE, B.Sc., F.R.A.S., Superintendent of the Kew Observatory*.

It is not many years since the operation of freezing mercury was considered a scientific feat, and a party of gentlemen were invited to the Kew Observatory to assist at the performance of the operation by the late Mr. R. Adams. That gentleman produced the solidified mercury by the action of ether upon carbonic acid in the solid form, and with his assistance considerable quantities of mercury were frozen, and a number of observations were made with carefully constructed thermometers, which gave the value for the melting-point of mercury which has since been accepted as the fixed point. Prof. Balfour Stewart communicated an account of these experiments to the Royal Society, in whose publications they will be found (Phil. Trans. 1863, pp. 425-435).

Since that date the determination of the low-temperature fixed point of thermometers has become a regular operation in the verification department of the Kew Observatory, and some eight or nine dozen instruments are compared in melting mercury annually, most of them being destined for use in Canada and other cold countries.

As the mode of making this comparison is not described in any readily accessible publication, we have frequently been requested to give an account of it, and have accordingly drafted this paper with a view of putting it on record.

The carbonic acid in its liquid form is purchased generally of Orchard, of High Street, Kensington, in the well-known

* Read November 14, 1885.

heavy iron bottles so familiar in lecture-rooms as recipients of compressed oxygen. A convenient quantity for the prosecution of an experiment is 200 gallons of gas, which weighs about 3 lb. The quantity of mercury operated upon is from 17 to 20 lb. By means of a common salt and ice freezing-mixture, the vessels, thermometers, mercury, and ether are all cooled down to about -10° C. beforehand.

The collecting-box being applied to the nozzle of the gas-receiver, the valve is unscrewed and gas allowed to escape through the apertures in the handles of the box for about a minute, during which time about 550 grains of solid carbonic gas are deposited in the form of snow.

The box being opened, the snow is removed with an ivory paper-knife and laid on the surface of the mercury previously placed in a wooden box well jacketed with felt. 2 oz. of ether is then poured over it, and the pasty mass forced under the surface of the mercury by means of a special wooden stirrer. The liberated gas and ether-vapour escaping through the mercury rapidly cool its temperature down, and a thin film of solid mercury forms on the top of the mass, which is immediately broken up by the stirrer. About six charges of acid snow and six fluid ounces of ether are usually necessary to convert the whole of the mercury into the finely granular mass of a pasty consistence.

It is considered inadvisable to freeze the mercury into a solid state, as then it becomes necessary to take it out of the box, and cut it up with shears into small lumps before the thermometers can be inserted into it, an operation which entails great loss of time and frozen mercury.

When the mass is in the semi-solidified condition the thermometers are plunged into it and allowed to fall to the requisite temperature—a well-verified standard being always placed in the bath at the same time and carefully watched during the progress of the comparison.

As soon as this standard is seen to begin to rise above the lowest point, the thermometers are taken out, and an additional quantity of snow and ether forced into the mercury until its temperature is again reduced, and the *feel* of the thermometer shows it has regained the proper consistency.

When the operation is properly and carefully performed, it

is found that the work of comparison may be spread over from half an hour to forty minutes, a period which permits of more than 100 mercurial thermometers being read with the necessary exactitude. If the thermometers are ordinary alcohol or Rutherford minimums with large bulbs, only from thirty to forty can be compared at a time.

The operation might be continued for an indefinite period were a sufficient supply of liquefied carbonic acid available, but at Kew we usually find the 200-gallon bottle, which, as already referred to, contains about 3 lb. weight of gas at sufficient pressure to afford snow, adequate for our purpose.

The cost of the materials is as follows:—gas 19s. 2d; ether 1s. 2d.

We do not wish to claim any credit for the process above described, but only to bring before the Members of the Society a description of our *modus operandi* as a matter which may perhaps be of a little interest to some of them.

XXXII. *Note on the Calibration of Galvanometers by a Constant Current.* By T. MATHER, Assistant in the Physical Department, City and Guilds of London Central Institution*.

THE principle of the method to be described consists in measuring the turning moment exerted on the magnetic needle of the galvanometer by a constant current, when the needle and coils are in different relative positions.

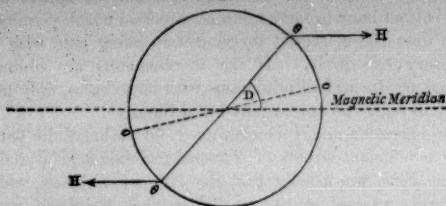
Suppose the galvanometer is placed in any position in a uniform field, and that a suitable current be passed through it, which current we will take as our *unit* current. The needle will take up some position of equilibrium, say, at an angle D to the magnetic meridian, and at angle θ to the zero of the scale (fig. 1).

When in this position, the controlling couple is $::$ to $\sin D$; and since this is balanced by the deflecting couple due to unit current, we see that this couple due to unit current, when the needle points to θ is also $::$ to $\sin D$ (fig. 1).

But in the ordinary use of the instrument the line of zeros

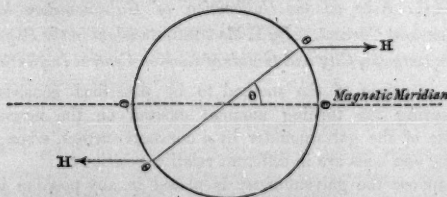
* Read November 28, 1885.

Fig. 1.



is in the magnetic meridian, and when the needle is deflected through angle θ the controlling couple is $\therefore$ to $\sin \theta$. Hence the current required to produce this deflection θ when the galvanometer is used in the ordinary way, is $\frac{\sin \theta}{\sin D}$. For the currents must be proportional to the controlling couples, since the relative positions of the needle and coils are the same (fig. 2).

Fig. 2.



This being true for any deflection, the calibration-curve is got by plotting θ and $\frac{\sin \theta}{\sin D}$.

Of course the same strength of current must be used in determining the different values of D .

To measure the angle D corresponding to any θ , we have merely to break the circuit and observe the angle through which the needle moves before coming to rest in the magnetic meridian. θ is read off directly on the scale of the galvanometer when the current is passing.

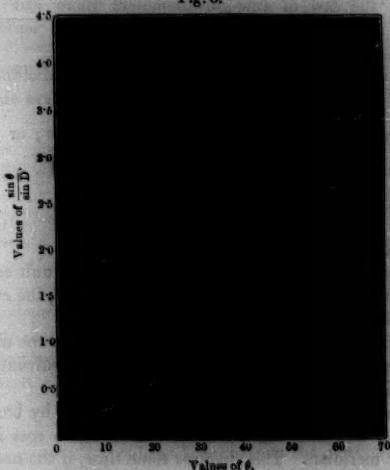
Below are appended the results of a calibration obtained by

this method, for which I have to thank Mr. Kilgour, a student of the Central Institution, who made the observations :—

θ .	D.	$\frac{\sin \theta}{\sin D}$.
10	34.8	.304
20	35.3	.592
30	32.2	.938
40	29.2	1.318
50	25.0	1.813
60	19.2	2.633
70	12.9	4.209

If θ and $\frac{\sin \theta}{\sin D}$ be plotted, we get the calibration-curve for that particular galvanometer (fig. 3).

Fig. 3.



In the foregoing Table the values of θ are integral numbers of degrees; but this is not at all necessary, for no fine adjustment, such as is required with sine-galvanometers, need be attempted. All that is needed is that the position of the needle, as indicated on the scale of the galvanometer, should

be accurately noted before breaking the circuit to observe the corresponding D .

The method described above may be utilized as a ready means of testing whether a galvanometer, whose law is *supposed to be known*, does *actually* conform thereto. For if the law of the galvanometer is $C=f(\theta)$, then, on plotting $f(\theta)$ and $\frac{\sin \theta}{\sin D}$, a straight line should result.

In the case of tangent-galvanometers, where $f(\theta) = k \tan \theta$ (k being a constant), we have to plot $\frac{\sin \theta}{\sin D}$ and $\tan \theta$ (i. e. $\frac{\sin \theta}{\cos \theta}$). Thus $\sin \theta$ is common to both variables; and hence, if we plot $\sin D$ and $\cos \theta$, we should get a straight line if the galvanometer be really a tangent one.

About the law of sine-galvanometers there can be no question if the controlling field be uniform; and herein lies one great advantage of sine-galvanometers.

In "proportional" galvanometers, where the deflection is supposed to be proportional to the current, the curve obtained by plotting θ and $\frac{\sin \theta}{\sin D}$ should be a straight line; or

$$\frac{\sin \theta}{\theta \sin D} = \text{a constant},$$

if calculation be preferred.

It may be pointed out that, by choosing our unit current such that the maximum value of D is nearly 90° , the method admits of great sensibility.

The quantity of apparatus required to perform the calibration is reduced to a minimum, nothing but the galvanometer and a suitable current-generator being needed.

As described above, the angle D is measured by breaking the circuit and allowing the needle to come to rest in the magnetic meridian. This requires some time, if the needle is not dead-beat, and the observations could be greatly facilitated by the use of a fixed divided circle or pointer, by which D could be read off simultaneously with θ without breaking the circuit. If so arranged, the whole operation could be completed in a few minutes; but as the method is chiefly

characterized by the fewness and simplicity of the apparatus required, it was thought undesirable to mar that simplicity by the addition of something not essential.

In conclusion, I desire to thank Professor Ayrton, F.R.S., for his kindness in suggesting the simple way of describing the method adopted in this note, and also for the statement of a problem which led to the results herein stated.

XXXIII. *Some Thermodynamical Relations.*—PART I.

By WILLIAM RAMSAY, *Ph.D.*, and SYDNEY YOUNG, *D.Sc.**

THE relations to be considered in the following pages are, we believe, well founded; but we hope to confirm them by more exact experiments than have as yet been made.

It appears advisable here to state the share which each of the authors has had in this work. For our purpose, the equation

$$\frac{L}{s_1 - s_2} = \frac{dp}{dt} \cdot \frac{t}{J}$$

will be employed. The relations of the first term of this equation were the subject of a communication to the Chemical Section of the Philosophical Society of Glasgow in the year 1877 by Dr. Ramsay, which, however, he considered too incomplete to be published; while the application of the last term of the equation to the vapour-pressures of substances, which formed the subject of a joint research by both authors, was independently discovered by Dr. Young. It is also right here to mention that some of the relations discovered by Dr. Ramsay have been pointed out subsequently and independently by Trouton (*Phil. Mag.* 1884, vol. xviii. p. 54).

The term $\frac{L}{s_1 - s_2}$ (where L represents the heat of vaporization of a liquid or solid substance at its boiling- or volatilizing-point, and where s_2 is the volume of the liquid or solid at these temperatures, and s_1 that of the gas into which one or other is converted), if stated in words, denotes the heat

* Read November 28, 1885.

expressed in units required to produce unit increase of volume of substance at the temperature of ebullition of the liquid or volatilization of the solid.

Two laws have been discovered, representing certain relations between different liquids.

The first law may be stated thus :—*The amount of heat required to produce unit increase of volume in the passage from the liquid to the gaseous state at the boiling-point under normal pressure is approximately constant for all bodies; or* $\frac{L}{s_1 - s_2} = C$.

The data on which this law is based are imperfect. The heats of vaporization of but few bodies have been determined with accuracy; and we have recently shown in a communication to the Royal Society that the results obtained by Favre and Silbermann, by Regnault, and by Andrews for ethyl alcohol are only approximations to truth. It is probable that the data given for other substances less easy to obtain in a pure state are still less to be relied on. We have also shown in that paper that the density of saturated vapour of alcohol is normal, or nearly so, only at temperatures below 50° . Nevertheless, in calculating by means of the above-mentioned formula the constant for alcohol, it has been assumed that the saturated vapour of alcohol possesses normal density at $78^\circ.4$; and from want of knowledge it has similarly been assumed that at their boiling-points the saturated vapours of the other liquids to be considered have also normal density. It must therefore be acknowledged that considerable doubt rests on both the expressions in the term $\frac{L}{s_1 - s_2}$, and this doubt prevented the previous publication of these relations; but it will be seen in the sequel that the much more trustworthy determinations of vapour-pressures amply confirm the law which has been stated.

The following table exhibits these relations at normal pressure:—

Substance.	$s_1 - s_2$.	L.	$\frac{L}{s_1 - s_2}$.
Water	1695	537	0.3168
Methyl alcohol	865.9	263.7	0.3046
Ethyl alcohol.....	624.7	202.4	0.3240
Ethyl oxide	322.3*	90.5	0.2808
Methyl formate.....	417.1	117.1	0.2808
Methyl acetate	362.5	110.2	0.3040
Ethyl acetate.....	323.0	92.7	0.2870
Ethyl oxalate	256.2	72.7	0.2835
Amyl alcohol (?)	375.5	121.4	0.3238
Acetic acid.....	318.0†	101.9	0.3204
Benzene	370.6	92.3	0.2491
Bromine.....	169.2	45.6	0.2695
Phosphorus chloride..	209.5	51.4	0.2453
Carbon disulphide ..	343.3	86.7	0.2526
Methyl iodide	181.5	46.1	0.2539
Ethyl iodide	180.5	46.9	0.2598
Ethyl chloride	361.9	100.1	0.2767
Carbon tetrachloride .	186.3	46.9	0.2515
Chloroform	229.1	61.2	0.2673
Mercury.....	258.0	77.5	0.3004
Nitric peroxide	309.6‡	93.48	0.3019

It will be seen that these numbers vary between 0.2453 for phosphorus chloride and 0.3240 for ethyl alcohol. The alcohols, water, and acetic acid give all nearly 0.32; and benzene, and bodies containing halogens and sulphur give lower numbers, averaging 0.2564.

The sequel will show that these numbers cannot be regarded as experimental deviations from a constant; but that they have significance can hardly be denied. It would also appear that the nature of the element in the compound, and its amount influence the absolute value; though the nature and extent of this influence can hardly be formulated.

The second law is:—*If the amounts of heat required to produce unit increase of volume in the passage from the liquid to the gaseous state be compared at different pressures for any two bodies, then the ratio of the amount at the boiling-point under a pressure p_1 , to the amount at another pressure p_2 , is approximately constant for all liquids.*

Or, $\frac{L}{s_1 - s_2}$ at p_1 , bears a constant proportion to $\frac{L}{s_1 - s_2}$ at p_2 , for all liquids, and probably for all solids.

* Determined by Horstmann, *Annalen*, 1868, Suppl. vi. p. 64.

† Determined by Ramsay and Young.

‡ Density taken as 39 ($H=1$) as an approximation.

In support of this statement the facts to be adduced are as follows :—

Water.					Chloroform.			
P. mms.	T.	$s_1 - s_2$ ces.	L.	$\frac{L}{s_1 - s_2}$	T.	$s_1 - s_2$ ces.	L.	$\frac{L}{s_1 - s_2}$
300	76.4	4022	552.1	0.137	35.0	535	64.65	0.121
760	100.0	1695	536.5	0.317	61.0	229	61.2	0.267
1000	107.1	1313	531.5	0.405	69.3	178	60.4	0.339
2000	129.3	695	515.6	0.742	92.5	95.3	58.2	0.611
3000	143.3	479	505.7	1.055	108.5	66.3	56.6	0.854
4000	154.2	369	497.7	1.350	121.0	51.4	55.4	1.079
5000	162.8	301	491.6	1.633	131.0	42.1	54.5	1.294
6000	170.1	255	486.1	1.906	140.0	35.9	53.6	1.493
7000	177.0	222	481.3	2.168	148.0	31.4	52.8	1.684
8000	182.9	197	477.0	2.424	155.0	27.9	52.2	1.871

The ratios, placing $\frac{L}{s_1 - s_2}$ at 300 millims. pressure = 1.0 for each substance, are :—

P	300	760	1000	2000	3000	4000	5000	6000	7000	8000
Water	1.00	2.30	2.94	5.40	7.68	9.81	11.88	13.86	15.77	17.63
Chloroform...	1.00	2.22	2.81	5.06	7.07	8.94	10.72	12.38	13.96	15.51

Another method of comparison is to regard the value of $\frac{L}{s_1 - s_2}$ for water as equal to unity at each pressure, when the numbers for chloroform become

1.00 0.902 0.953 0.938 0.922 0.911 0.903 0.893 0.885 0.850.

Similar results may be obtained on comparing other bodies with water; it may be, however, remarked that the instance chosen is one of those in which least concordance is to be noticed. It is evident that here also the concordance is merely approximate.

Another point worthy of notice is this. The heat of vaporization is expended in at least two channels :—it produces expansion against pressure, thus doing external work ; and it is partly expended in internal work on the molecules of the body. Now it follows from what has preceded that the internal and total work bear an approximately constant proportion to each other for any one pressure, whatever be the liquid. Thus at a pressure of approximately 760 millims. the ratio of external to total work is shown as follows :—

Substance.	Total work. External work.	Substance.	Total work. External work.
Bromine	11.09	Ethyl acetate	11.81
Phosphorus chloride ...	10.09	Ethyl oxalate	11.67
Carbon disulphide	10.39	Amyl alcohol	13.32
Methyl alcohol	12.53	Ethyl chloride	11.39
Ethyl alcohol	13.33	Benzene	10.25
*Ethyl oxide (ether)	11.55	Chloroform	10.99
Water	13.02	Carbon tetrachloride ...	10.35
Methyl iodide	10.45	Mercury	12.36
Ethyl iodide	10.69	*Nitric peroxide	12.42
Methyl formate	11.55	Acetic acid	13.18
Methyl acetate	12.51	Water	13.02

* Vapour-density found at a few degrees above boiling-point.

In the equation $\frac{L}{s_1 - s_2} = \frac{dp}{dt} \cdot \frac{t}{J}$, both terms are approximately known for a number of substances. But before comparing them it is necessary to reduce the pressures which have been given in the previous part of this paper in millimetres of mercury, to grams per square centimetre. The comparison has been made for water, with the following results:—

Pressures, in millim. of mercury.	$\frac{L}{s_1 - s_2}$	$\frac{dp}{dt} \cdot \frac{t}{J}$
300 . . .	0.137	0.1395
1000 . . .	0.405	0.414
2000 . . .	0.742	0.770
3000 . . .	1.055	1.110
5000 . . .	1.633	1.743

The non-equality of these numbers may be ascribed to various causes: want of accuracy in experiment, most probably in determination of the heats of vaporization, and the assumption that the density of the saturated vapour of water is normal at high pressures, are among the most likely. With such results for water, it was thought unnecessary to compare these terms for other liquids, where the constants are probably less accurately determined. These numbers, however, suffice to prove that the relations pointed out in the first part of this paper, although derived from observations and assumptions not always trustworthy, agree fairly well with those deduced from the vapour-pressures, which have been determined with a high degree of accuracy.

We now proceed to exhibit the relations between the vapour-pressures of various liquids. The constants of the bodies considered were determined by Regnault, Olszewski, Naumann, Isambert, Moitessier and Engel, and by ourselves.

If a curve be constructed to represent the relation of temperature to pressure for any substance, and if tangents be drawn to touch the curve at various points corresponding to certain temperatures, these tangents will give the rate of increase of pressure per unit rise of temperature; in other words, the value $\frac{dp}{dt}$ for those temperatures.

If we construct curves for a number of substances, and determine the value of $\frac{dp}{dt}$ for each of them at the same temperature, it is clear that the values obtained will differ widely, and will be greater for volatile substances than for those which are less volatile. But if we determine the values of $\frac{dp}{dt}$ for the same series of bodies, not at the same temperature, but at the same pressure, the conditions under which the comparison is made will be more similar, and the resulting values may be expected to differ much less.

In the calculation of the vapour-pressures of a number of substances for each degree between certain limits of pressure, it became evident that at any given pressure the rate of increase was generally, though not always, greater for the volatile substances than for the less volatile.

It was found that the product of the absolute temperature into the rate of increase of pressure $\left(\frac{dp}{dt} \cdot t\right)$ at any given pressure was approximately the same for the bodies examined, but the differences were evidently too great to be ascribed to errors of experiment or of calculation. That this product should be approximately the same for different substances might perhaps be anticipated from the following considerations:—If there are two bodies, the absolute temperatures of which must be raised to 200° and 400° respectively, in order to produce a certain effect, the same for both, it might be expected that a further rise of temperature of 1° would produce a greater effect on the substance whose temperature was

200° than on that at 400°; for the rise of temperature in the first case is greater in proportion to the temperature to which the body has already attained than it is in the second, the rise in the one case being from 200° to 201°, and in the other from 400° to 401°. The rise of temperature would perhaps rather be proportional if the temperature of the hotter body were raised from 400° to 402°, or if the rise of temperature in each case were made proportional to the absolute temperature. In order, then, to make the conditions as similar as possible in the calculation of the value of $\frac{dp}{dt}$ for different substances, the magnitude of the unit degree of temperature should be made to vary in the same ratio as the absolute temperature of those substances, corresponding to the particular pressure at which they are compared; or, in other words, keeping the unit degree constant, the value $\frac{dp}{dt}$ should be multiplied by the absolute temperature t .

The values of $\frac{dp}{dt} \cdot t$ were determined for a number of liquids at several different pressures; and it was found that the products obtained for different stable substances at the same pressure were always approximately the same, whatever that pressure might be. Thus at a pressure of 400 millims. the following values were obtained:—

Carbon bisulphide	4436	1·000
Alcohol	5873	1·324
Chlorobenzene	4724	1·065
Bromobenzene	4703	1·060
Aniline	5124	1·155
Methyl salicylate	4959	1·112
Bromonaphthalene	4930	1·111
Mercury	4812	1·085

This may be better seen by making one of these values equal to unity, and reducing the others in the same ratio. The second column is thus calculated.

Repeating this operation at another pressure, a second series of values was obtained; and these were reduced in the same

way. It was at once noticed that the reduced numbers at the new pressure were identical within the limits of error of experiment and calculation with those at the first pressure.

The calculations were therefore continued so as to include the widest attainable ranges of pressure, and at every pressure the value of $\frac{dp}{dt} \cdot t$ for one substance was made equal to unity; and the others reduced in the same ratio. The values of $\frac{dp}{dt} \cdot t$ for water were made = 1, because the vapour-pressures of this substance have been determined by Regnault between wide limits of temperature with very great care; but it was noticed that the values for both water and alcohol were very much higher than for the other substances examined; and a similar comparison was therefore made by taking the values for carbon bisulphide as equal to unity.

It was then found that for pressures ranging between 150 and 1500 or 2000 millim. the reduced values for each substance, with the exception of mercury, were very nearly constant for all pressures.

In the case of alcohol and water, employing the vapour-pressures of water as determined by Regnault (*Mémoires de l'Académie*, vol. xxv.), and of alcohol as determined by Ramsay and Young, the ratio of the values $\frac{dp}{dt} \cdot t$ was seen to be the same at pressures between 150 and 20,000 millim.

Two methods of calculation were adopted. In the first and more accurate method the vapour-pressures for each degree were calculated by the method of differences; and from these numbers the value of $\frac{dp}{dt}$ at any pressure could easily be calculated with very small error. The time required by this process is very great; and it was only adopted in the case of water, for which the pressures for each degree between -32° and 230° have already been calculated (Balfour Stewart, 'Treatise on Heat'), and of carbon bisulphide, alcohol, chlorobenzene, bromobenzene, aniline, methyl salicylate, bromonaphthalene, and mercury between certain limits of pressure, generally 150 or 200 to 700 millim. (Ramsay and Young, *Chem. Soc. Journ.* 1885, p. 640).

In all other cases tangents were drawn to touch the vapour-pressure curves at points corresponding to definite pressures, several curves being required for each substance on different scales to admit of this being done. With care, this method yields fairly satisfactory results.

It has been mentioned that the identity of the reduced numbers for any one substance holds at pressures ranging between about 150 and 1500 to 2000 millim. At lower pressures it is much more difficult to determine the values of $\frac{dp}{dt}$

with accuracy, and the influence of experimental errors is greater. Moreover, it has been necessary provisionally to adopt the method of tangents for all substances except water and carbon bisulphide at low pressures; and it is doubtful, therefore, whether much confidence is to be placed in the values at pressures below 150 millim.

As regards high pressures, with the exception of alcohol all the substances have been investigated by Regnault; and it may be worth while to mention one or two facts which appear to throw some little doubt on the accuracy of some of his determinations at pressures above 2000 or 3000 millim. In vol. xxvi. of the *Mémoires de l'Académie* Regnault gives the vapour-pressures of a large number of substances for each 5°, calculated from his empirical formulæ. Many of these series have been examined by the method of differences; and it has been found that in several cases the numbers forming the third set of differences increase slowly up to pressures of from 2000 to 3000 millim., but decrease at higher pressures. This is notably the case with alcohol; and it has been shown by Ramsay and Young, in a paper read before the Royal Society in May 1885, that the higher vapour-pressures of this substance as determined by Regnault are too low. It has also been pointed out by Vincent and Chappuis (*Compt. Rend.* vol. c. p. 1216) that the vapour-pressures of methyl chloride above 7 atmospheres are much higher than those calculated from Regnault's formula, the difference at 140° amounting to no less than 11.3 atmospheres.

Where this decrease is observable in the third set of differences, it seems therefore justifiable to doubt the accuracy of the results at the higher pressures.

It may be stated in conclusion:—(1) That the values of $\frac{dp}{dt} \cdot t$ are approximately the same for all stable substances at the same pressure, but that the differences are real, and are not due to errors of experiment or calculation; and (2) that the rate of increase of this value $\frac{dp}{dt} \cdot t$ with rise of pressure is the same for all stable bodies, at any rate for pressures between 150 and 2000 millim., while for alcohol and water it is the same for all pressures between 150 and 20,000.

In the tables which follow, the values of $\frac{dp}{dt} \cdot t$ (absolute temperature), and $\frac{dp}{dt} \cdot t$ are given for a number of stable substances at definite pressures, also the reduced values of $\frac{dp}{dt} \cdot t$, that for (1) water and (2) carbon bisulphide being made = 1.000 at each pressure, the values for other substances at the same pressure being reduced in the same ratio.

The second series of tables contains similar data for several substances which dissociate more or less completely on their passage into the gaseous state. It will be seen that for such bodies the values of $\frac{dp}{dt} \cdot t$ at any pressure are considerably higher than for stable substances at the same pressure.

The initial letter of the name of the observer of the vapour-pressure of each substance is given in the table containing the values of $\frac{dp}{dt}$ at the foot of the vertical columns. R. stands for Regnault, R. and Y. for Ramsay and Young, O for Olszewski, N. for Naumann, I. for Isambert, M. & E. for Moitessier and Engel.

Values of $\frac{dp}{dt}$ for various Stable Substances at Definite Pressures.

Pressure in millims.	Carbon bisulphide. CS_2 .	Ethyl iodide. $\text{C}_2\text{H}_5\text{I}$.	Ethyl bromide. $\text{C}_2\text{H}_5\text{Br}$.	Ethyl chloride. $\text{C}_2\text{H}_5\text{Cl}$.	Carbon tetrachloride. CCl_4 .	Ether. $\text{C}_4\text{H}_{10}\text{O}$.	Bromobenzene. $\text{C}_6\text{H}_5\text{Br}$.	Chlorobenzene. $\text{C}_6\text{H}_5\text{Cl}$.	Chloroform. CHCl_3 .	Mercury. Hg.	Ethylene. C_2H_4 .	Sulphur. S.	Bromonaphthalene. $\text{C}_{10}\text{H}_7\text{Br}$.	Methyl salicylate. $\text{C}_9\text{H}_8\text{O}_3$.	Aniline. $\text{C}_6\text{H}_5\text{NH}_2$.	Methyl alcohol. CH_3OH .	Water. H_2O .	Ethyl alcohol. $\text{C}_2\text{H}_5\text{OH}$.
10	2.66	4.52	2.86	2.57	2.57	2.13	2.13	2.13	0.302	1.31	4.54	1.43	1.93	2.00	0.738	0.66	0.66	0.66
50	2.66	4.52	2.86	2.57	2.57	2.13	2.13	2.13	0.302	1.31	4.54	1.43	1.93	2.00	0.738	0.66	0.66	0.66
100	4.78	6.28	5.19	4.62	4.62	3.83	3.83	3.83	2.47	2.47	6.67	2.06	3.40	3.77	5.48	4.05	4.05	4.05
150	6.65	8.00	7.15	6.30	6.30	5.19	5.19	5.19	3.46	3.46	12.41	2.41	4.12	4.65	6.38	4.65	4.65	4.65
200	8.38	8.00	8.80	8.07	8.07	6.59	6.59	6.59	4.40	4.40	15.97	2.41	5.24	6.05	6.76	5.44	5.44	5.44
300	11.70	11.08	12.25	13.45	10.69	9.24	9.24	9.82	8.64	8.64	22.67	3.07	7.32	8.30	11.80	13.40	12.50	12.50
400	14.75	14.08	15.47	16.94	13.88	11.60	11.60	12.84	15.25	15.25	30.07	3.71	9.40	10.55	14.10	20.56	19.10	17.48
500	17.65	17.65	18.86	20.08	16.90	13.85	13.85	14.68	18.16	17.78	37.78	4.37	11.35	12.65	16.30	25.71	22.20	21.10
600	20.38	20.38	21.65	22.90	19.50	15.90	15.90	17.00	21.00	21.44	45.00	5.00	13.10	14.65	18.40	30.63	25.30	24.80
700	23.10	23.10	24.39	26.91	21.47	18.05	18.05	19.10	23.76	23.05	52.00	5.62	14.75	16.50	20.40	35.42	31.46	31.46
800	25.67	25.67	27.33	29.17	23.94	20.35	20.35	21.40	26.32	26.00	59.00	6.25	16.35	18.40	22.40	40.00	36.00	36.00
900	28.60	28.60	29.70	32.47	26.28	22.65	22.65	23.70	28.70	28.00	66.00	6.87	18.00	20.00	24.40	44.00	40.00	40.00
1000	30.80	30.80	32.45	35.12	28.33	24.95	24.95	26.00	31.74	31.65	73.00	7.50	19.65	21.65	26.40	50.00	44.00	44.00
1500	42.43	42.43	44.25	48.25	38.28	45.11	45.11	46.25	42.78	42.60	80.00	8.12	21.30	23.30	28.40	60.00	50.00	50.00
2000	52.40	52.40	55.00	60.40	48.75	58.54	58.54	59.68	52.98	52.80	87.00	8.75	23.00	25.00	30.40	70.00	60.00	60.00
3000	70.67	70.67	75.10	83.80	65.90	78.00	78.00	79.14	72.94	72.80	94.00	9.38	24.65	26.65	32.40	80.00	70.00	70.00
5000	105.00	105.00	108.9	119.80	96.62	116.00	116.00	104.70	104.70	104.60	101.00	10.00	26.30	28.30	34.40	90.00	80.00	80.00
10,000																		
15,000																		
20,000																		

1000 mm.)
and
R. & Y.
(above
1000 mm.)

Values of $\frac{dp}{dt}$. (where t is the absolute temperature) for various Stable Substances at Definite Pressures.

Pressure in mm.	Carbon bisulphide. CS ₂ .	Ethyl iodide. C ₂ H ₅ I.	Ethyl bromide. C ₂ H ₅ Br.	Ethyl chloride. C ₂ H ₅ Cl.	Carbon tetrachloride. CCl ₄ .	Ether. C ₄ H ₁₀ O.	Bromobenzene. C ₆ H ₅ Br.	Chlorobenzene. C ₆ H ₅ Cl.	Chloroform. CHCl ₃ .	Mercury. Hg.	Ethylene. C ₂ H ₄ .	Sulphur. S.	Bromo- naphthalene. C ₁₀ H ₇ Br.	Methyl salicy- late. C ₉ H ₉ O ₃ .	Aniline. C ₆ H ₅ NH ₂ .	Methyl alcohol. CH ₃ OH.	Water. H ₂ O.	Ethyl alcohol. C ₂ H ₅ OH.	
10	676.7		715		722		740			136	176					191	184.8	183	
50										664	634					860	840.5	863	
100	1280	1315	1364		1333	1304	1306	1384		1320	1419					1617	1607	1642	
150	1839	1884	1938	1900	1915	1928	1946	1992		1903	1879					2251	2315	2403	
200	2373	2436	2438	2528	2510	2531	2525	2558	2576	2483	2474					2947	3016	3155	
300	3439	3504	3513	3536	3537	3566	3657	3670	4772	3646	3627					4256	4361	4541	
400	4436	4566	4647	4664	4680	4732	4703	4724	5829	5948	4920					5560	5690	5873	
500	5417	5654	5620	5684	5612	5724	5728	5728	6829	7067						6770	6906	7197	
600	6361	6597	6607	6697	6649	6723	6678	6740	7862	8168						7917	8136	8567	
700	7317		7537	7624	7446	7673	7688	7678	8812	9243						8228	8379	8945	
800	8235		8554	8672	8403	8539	8602		9709	10173						9229	9379	9846	
900	9281		9394	9420	9330	9659			10846	13801						11441	11825	11094	
1000	10102		10371	10290	10170	10613			15191	15463						12585	12565	12565	
1500	14507		14726	14716	14470	14819			19884							18131	13949	13827	
2000	18469		18848	19141	18783	19787			27813							24004	18667	19718	
3000	28006		29886	29564	28783	27730			42320							33066	24070	25179	
5000	41128		41306	41702	41344	43314										53440	34736	35970	
10,000																	54482	56480	58480
15,000																	100250	108670	109670
20,000																	142680	148480	150770

Reduced Values of $\frac{dp}{dt}$, that for Water being made = 1.000 at each pressure, and the values for other substances at the same pressure reduced in the same ratio.

Pressure in millims.	Carbon bisulphide. CS_2 .	Ethyl iodide. $\text{C}_2\text{H}_5\text{I}$.	Ethyl bromide. $\text{C}_2\text{H}_5\text{Br}$.	Ethyl chloride. $\text{C}_2\text{H}_5\text{Cl}$.	Carbon tetrachloride. CCl_4 .	Ether. $\text{C}_4\text{H}_{10}\text{O}$.	Bromobenzene. $\text{C}_6\text{H}_5\text{Br}$.	Chlorobenzene. $\text{C}_6\text{H}_5\text{Cl}$.	Chloroform. CHCl_3 .	Mercury. Hg.	Ethylene C_2H_4 .	Sulphur. S.	Bromo- naphthalene. $\text{C}_{10}\text{H}_7\text{Br}$.	Methyl salicylate. $\text{C}_8\text{H}_9\text{O}_3$.	Anilino. $\text{C}_6\text{H}_5\text{NH}_2$.	Methyl alcohol. CH_3OH .	Water. H_2O .	Ethyl alcohol. $\text{C}_2\text{H}_5\text{OH}$.
10	804	819	851	830	830	811	880	861	854	736	951	865	865	894	892	1.034	1.000	991
50	796	819	849	830	830	811	868	861	854	790	764	863	865	877	892	1.024	"	1.014
100	794	814	837	821	837	833	840	856	854	821	812	863	865	877	890	1.007	"	1.022
150	794	814	837	821	837	833	840	856	854	821	812	863	865	877	890	1.007	"	1.022
200	787	804	806	838	832	839	837	848	854	823	832	861	863	894	894	1.007	"	1.022
300	787	804	806	838	832	839	837	848	854	823	832	861	863	894	894	1.007	"	1.022
400	784	810	803	833	833	840	831	835	844	850	839	861	863	874	894	1.007	"	1.022
500	784	810	803	833	833	840	831	835	844	850	839	861	863	874	894	1.007	"	1.022
600	782	811	818	823	833	840	829	828	842	860	839	861	863	874	894	1.007	"	1.022
700	780	811	818	823	833	840	829	828	842	860	839	861	863	874	894	1.007	"	1.022
800	783	813	813	813	794	811	818	819	838	878	865	865	864	871	891	1.007	"	1.022
900	785	813	813	813	794	811	818	819	838	878	865	865	864	871	891	1.007	"	1.022
1000	785	813	813	813	794	811	818	819	838	878	865	865	864	871	891	1.007	"	1.022
1500	781	813	813	813	794	811	818	819	838	878	865	865	864	871	891	1.007	"	1.022
2000	767	813	813	813	794	811	818	819	838	878	865	865	864	871	891	1.007	"	1.022
3000	750	813	813	813	794	811	818	819	838	878	865	865	864	871	891	1.007	"	1.022
6000	755	813	813	813	794	811	818	819	838	878	865	865	864	871	891	1.007	"	1.022
10,000	755	813	813	813	794	811	818	819	838	878	865	865	864	871	891	1.007	"	1.022
20,000	755	813	813	813	794	811	818	819	838	878	865	865	864	871	891	1.007	"	1.022

Values of $\frac{dp}{dt}$ for various Dissociable Substances at definite Pressures.

Pressure in millim.	Nitrogen peroxide. N_2O_4	Ammonium chloride. NH_4Cl	Ammonium carbamate. $CO_2N_2H_6$	Chloral methyl-alcoholate. $CCl_3CH \begin{Bmatrix} OH \\ OCH_3 \end{Bmatrix}$	Chloral ethyl-alcoholate. $CCl_3CH \begin{Bmatrix} OH \\ OC_2H_5 \end{Bmatrix}$
10	...	...	0.99		
50	...	1.91	3.75	2.97	2.94
100	5.2	3.63	7.02	5.46	5.47
150	8.1	4.98	10.13	8.26	7.91
200	11.3	6.26	12.71	10.23	10.17
300	15.6	8.39	18.57	14.33	14.08
400	20.47	10.74	23.48	19.30	18.06
500	24.27	13.25	...		22.34
600	...	15.70	...	...	...
	R. and Y.	R. and Y.	N., I., and M and E.	R. and Y.	R. and Y.

Absolute Temperatures (t) of various Dissociable Substances corresponding to definite Vapour-pressures.

Pressure in millim.	Nitrogen peroxide. N_2O_4	Ammonium chloride. NH_4Cl	Ammonium carbamate. $CO_2N_2H_6$	Chloral methyl-alcoholate. $CCl_3CH \begin{Bmatrix} OH \\ OCH_3 \end{Bmatrix}$	Chloral ethyl-alcoholate. $CCl_3CH \begin{Bmatrix} OH \\ OC_2H_5 \end{Bmatrix}$
10	...	...	270.9		
50	...	524.7	289.9	331.2	334.0
100	256.0	542.8	299.4	343.3	346.1
150	263.1	554.6	305.1	350.8	353.7
200	268.1	563.3	309.8	356.1	358.1
300	275.9	577.3	316.1	364.2	367.4
400	281.4	587.9	321.0	370.3	373.8
500	285.9	596.0	...		378.7
600	...	603.0	...		...

Products of $\frac{dp}{dt}$ into Absolute Temperatures ($\frac{dp}{dt} \cdot t$) for various Dissociable Substances at definite Pressures.

Pressure in millim.	Nitrogen peroxide. N_2O_4	Ammonium chloride. NH_4Cl	Ammonium carbamate. $CO_2N_2H_6$	Chloral methyl-alcoholate. $CCl_3CH \begin{Bmatrix} OH \\ OCH_3 \end{Bmatrix}$	Chloral ethyl-alcoholate. $CCl_3CH \begin{Bmatrix} OH \\ OC_2H_5 \end{Bmatrix}$
10	...	...	268		
50	...	1002	1087	984	983
100	1331	1970	2102	1866	1893
150	2131	2762	3091	2898	2798
200	3030	3526	3938	3635	3642
300	4304	4844	5870	5219	5173
400	5787	6314	7537	7147	6751
500	6939	7897	...		8460
600	...	9467	...		...

Reduced Values of $\left(\frac{dp}{dt} \cdot t\right)$; that for Water being made = 1·000 at each pressure, and the values for other substances at the same pressure reduced in the same ratio.

Pressure in millim.	Nitrogen peroxide. N_2O_4 .	Ammonium chloride. NH_4Cl .	Ammonium carbamate. $CO_2N_2H_6$.	Chloral methyl-alcoholate. $CCl_3 \cdot CH \left\{ \begin{smallmatrix} OH \\ OCH_3 \end{smallmatrix} \right.$	Chloral ethyl-alcoholate. $CCl_3 \cdot CH \left\{ \begin{smallmatrix} OH \\ OC_2H_5 \end{smallmatrix} \right.$
10			1·451		
50		1·193	1·293	1·170	1·170
100	0·828	1·226	1·308	1·161	1·178
150	0·921	1·193	1·335	1·252	1·209
200	1·004	1·169	1·306	1·205	1·208
300	0·987	1·110	1·346	1·197	1·186
400	1·018	1·116	1·332	1·263	1·193
500	1·004	1·143			1·225
600		1·164			

Reduced Values of $\left(\frac{dp}{dt} \cdot t\right)$; that for Carbon-bisulphide being made = 1·000 at each pressure, and the values for other substances reduced in the same ratio.

Pressure in millim.	Nitrogen peroxide. N_2O_4 .	Ammonium chloride. NH_4Cl .	Ammonium carbamate. $CO_2N_2H_6$.	Chloral methyl-alcoholate. $CCl_3 \cdot CH \left\{ \begin{smallmatrix} OH \\ OCH_3 \end{smallmatrix} \right.$	Chloral ethyl-alcoholate. $CCl_3 \cdot CH \left\{ \begin{smallmatrix} OH \\ OC_2H_5 \end{smallmatrix} \right.$
10	...	...	...		
50	...	1·483	1·609	1·456	1·456
100	1·043	1·540	1·643	1·458	1·480
150	1·159	1·502	1·681	1·576	1·522
200	1·277	1·486	1·659	1·532	1·535
300	1·255	1·412	1·711	1·521	1·508
400	1·298	1·423	1·699	1·611	1·522
500	1·281	1·458	...		1·562
600	...	1·488			

A relation may be observed between the two substances bromobenzene and chlorobenzene, and also between ethyl chloride, ethyl bromide, and ethyl iodide. The ratio of the absolute temperatures of the bodies in either group corresponding to any given vapour-pressure is a constant. Thus the ratio of the absolute temperature of bromobenzene to that of chlorobenzene when the vapour-pressure of both is 100 millim. = $\frac{364 \cdot 3}{343 \cdot 5}$, or 1·061; and at the other pressures up to 700 millim. the ratio remains absolutely constant at 1·059.

The ratio of the absolute temperatures of ethyl bromide and ethyl chloride has been determined for pressures between 150 and 5000 millim., and the numbers found vary only between 1·089 and 1·091. The data for ethyl iodide are much less complete; the comparison with ethyl chloride can only be made at pressures between 150 and 500 millim., and with ethyl bromide between 50 and 500 millim. In the first case the ratio of the absolute temperatures varies between 1·206 and 1·209, and in the second case between 1·105 and 1·108.

PROCEEDINGS
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XXXIV. *Some Thermodynamical Relations.*—PART II.

By WILLIAM RAMSAY, *Ph.D.*, and SYDNEY YOUNG, *D.Sc.**

IN a note at the conclusion of the first of this series of papers, it was stated that the absolute temperatures of certain nearly related bodies corresponding to equal vapour-pressures are constant at all pressures. Further investigation, however, has shown that although it is only in the case of nearly allied substances, such as chlorobenzene and bromobenzene, that the ratio of the absolute temperatures corresponding to the same vapour-pressure is a constant, whatever (within the limits afforded by experimental data) that pressure may be, yet a relation does exist between the ratios of the absolute temperatures of all bodies, whether solid or liquid and whether stable or dissociable, which may be expressed in the case of any two bodies by the equation

$$R' = R + c(t' - t),$$

where R is the ratio of the absolute temperatures of the two bodies corresponding to any vapour-pressure, the same for both; R' is the ratio at any other pressure, again the same

* Read December 12, 1885.

for both; c is a constant which may be 0 or a small + or - number; and t' and t are the temperatures of one of the bodies corresponding to the two vapour-pressures.

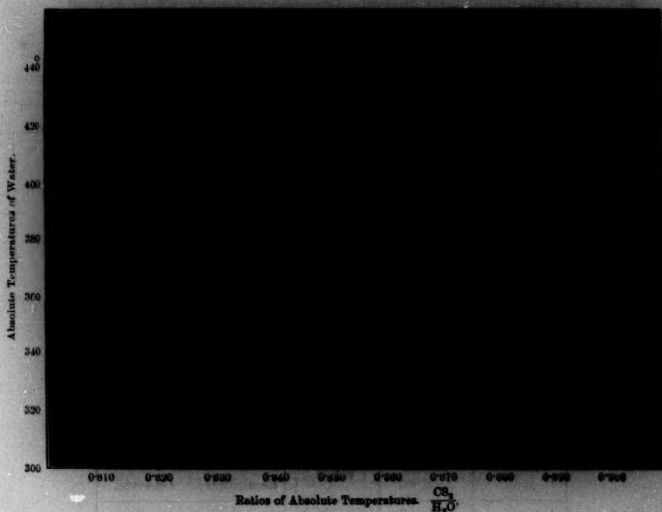
When $c=0$, $R'=R$, or the ratio of the absolute temperatures is a constant at all pressures; and where c is greater or less than 0, its value may readily be determined either by calculation, or graphically by representing the (absolute) temperatures of one of the two bodies as ordinates, and the ratio of the absolute temperatures at pressures corresponding to the absolute temperatures of that body as abscissæ. It is found that in all cases the points representing the relation of the ratio of the absolute temperatures of the two bodies to the absolute temperatures of one of them fall in a straight line. This is illustrated in some of the examples which are brought forward to prove the truth of the law. It follows from this that, if we know accurately the vapour-pressures of one substance, we only require two, or, better, three, accurate determinations of the vapour-pressure of any other substance, at temperatures moderately far apart, in order to be able to calculate the vapour-pressure of that substance at any required temperature, or, rather, to calculate the temperature corresponding to any vapour-pressure within the limits of pressure comprised in the determinations of the standard substance.

In the examples which follow, the same substance has not always been taken as the standard; but generally, in the comparison of any two bodies, that one is taken as correct which is likely to have been the more accurately investigated. Twenty-two different bodies have been examined, and they are arranged in twenty-three couples. The absolute temperatures corresponding to definite vapour-pressures of a number of substances are given in the tables in a previous portion of this paper. For other substances, curves were drawn to represent the relation of temperature to pressure, and the temperatures corresponding to definite pressures read off.

1. *Ratios of the Absolute Temperatures of Water and Carbon Bisulphide at Definite Vapour-pressures.*—As an example of the method employed, the whole of the data are given in this case, and also a diagram showing how the value of c may be obtained graphically. The absolute temperatures of water corresponding to the definite vapour-pressures are taken

to be correct, and are made the ordinates in fig. 1. The abscissæ are the ratios of the absolute temperatures of carbon bisulphide to those of water at those vapour-pressures. As

Fig. 1.



might be expected, some of the points representing the relation of the ratios of the absolute temperatures to the temperatures of water fall slightly to one or other side of the straight line drawn to pass through them. In order to find the error involved in these slight deviations from the straight line, points on the line itself corresponding to the same temperatures of water were read off, and these are given in the table as smoothed ratios. Lastly, the absolute temperatures of carbon bisulphide were calculated by multiplying those of water by the smoothed ratios; and the recalculated temperatures of carbon bisulphide are given together with the observed temperatures. It will be seen that the difference between the observed and recalculated temperatures is very small.

Pressures.	Absolute temperatures of water.	Ratios of absolute temperatures of CS_2 to those of H_2O .	Smoothed ratios.	Recalculated absolute temperatures of CS_2 .	Observed absolute temperatures of CS_2 .
millim.	°			°	°
50	311.3	.8161	.8160	254.0	254.05
100	324.7	.8245	.8242	267.6	267.7
150	333.1	.8301	.8296	276.3	276.5
200	339.6	.8339	.8338	283.2	283.2
300	348.9	.8403	.8400	293.1	293.2
400	356.0	.8448	.8449	300.8	300.75
500	361.7	.8485	.8483	306.8	306.9
600	366.5	.8517	.8519	312.2	312.15
700	370.7	.8545	.8545	316.8	316.75
800	374.45	.8567	.8571	320.9	320.8
900	377.8	.8589	.8590	324.5	324.5
1000	380.85	.8612	.8611	327.95	328.0
1500	393.2	.8695	.8692	341.8	341.9
2000	402.5	.8753	.8757	352.5	352.3
3000	416.5	.8852	.8850	368.6	368.7
5000	435.85	.8987	.8978	391.3	391.7

In the equation $R' = R + c(t' - t)$ the value of $c = .0006568$.

The greatest difference between the observed and recalculated temperatures is at the highest pressure (5000 millim.), and is only $0^{\circ}.4$.

2. *Water and Ethyl Alcohol.* $\frac{\text{H}_2\text{O}}{\text{C}_2\text{H}_5\text{O}}$ Temperatures of water taken as correct, and made the ordinates. $c = .0000962$.

Pressures.	Ratios.	Smoothed ratios.	Absolute temp. of Alcohol.	
			Recalculated.	Observed.
10	1.058	1.0527	270.1	269.25
50	1.0553	1.0553	295.0	295.0
100	1.0559	1.0567	307.3	307.5
150	1.0565	1.0574	315.0	315.3
200	1.0580	1.0580	321.0	321.0
300	1.0579	1.0589	329.5	329.8
400	1.0595	1.0596	336.0	336.0
500	1.0604	1.0602	341.2	341.1
600	1.0609	1.0606	345.55	345.45
700	1.0619	1.0610	349.4	349.25
800	1.0618	1.0614	352.8	352.65
900	1.0624	1.0617	355.8	355.6
1000	1.0626	1.0620	358.6	358.4
1500	1.0633	1.0632	369.8	369.8
2000	1.0648	1.0641	378.25	378.0
3000	1.0663	1.0655	390.9	390.6
5000	1.0675	1.0673	408.4	408.3
10000	1.0687	1.0701	435.15	435.7
15000	1.0701	1.0720	452.7	453.5
20000	1.0715	1.0735	466.2	467.1

The only noticeable differences between the observed and recalculated temperatures are at the three highest pressures and at 10 millim. With regard to the error at the low pressure, it may be mentioned that a number of careful determinations of the vapour-pressure of alcohol at low temperatures were made by Ramsay and Young by the still method (see note at end of paper), and that their results differed slightly from those of Regnault. At a pressure of 50 millim. the difference was only $0^{\circ}\cdot 05$, but at 10 millim. the absolute temperature found by them was $270^{\circ}\cdot 25$, which is practically the same as that recalculated by means of the equation

$$R' = R + c(t' - t).$$

3. *Ethyl Alcohol and Methyl Alcohol.* $\frac{C_2H_5O}{CH_3O}$. Temperatures of ethyl alcohol taken as correct. $c = \cdot 0001603$.

Pressures.	Ratios.		Absolute temperatures of Methyl Alcohol.	
	From observed temperatures.	Smoothed.	Recalculated.	Observed.
millim.				
10	<i>a</i> 1.0372	1.0475	256.5	259.0
10	<i>b</i> 1.0428	1.0473	257.9	259.0
50	1.0424	1.0433	282.8	283.0
100	1.0424	1.0414	295.3	295.1
150	1.0406	1.0400	303.2	303.0
200	1.0392	1.0392	308.9	308.9
300	1.0384	1.0379	317.75	317.6
400	1.0371	1.0368	324.1	324.0
500	1.0358	1.0359	329.3	329.3
600	1.0346	1.0351	333.75	333.9
700	1.0335	1.0346	337.4	337.8
800	1.0339	1.0340	341.05	341.1
900	1.0334	1.0336	344.05	344.1
1000		1.0331	346.9	
1500	1.0304	1.0314	358.5	358.9
2000	1.0297	1.0300	367.0	367.1
3000	1.0276	1.0280	380.0	380.1
5000	1.0256	1.0251	398.3	398.1

Two values are given for 10 millim.:—(*a*) the ratio calculated from Regnault's values, and (*b*) the ratio obtained by employing the recalculated temperature of ethyl alcohol. The only noticeable difference between the observed and recalculated temperatures is at a pressure of 10 millim.; and by taking the recalculated temperature of ethyl alcohol it is greatly reduced.

4. *Carbon Bisulphide and Ethyl Oxide.* $\frac{\text{CS}_2}{\text{C}_4\text{H}_{10}\text{O}}$. Temperatures of carbon bisulphide taken as correct. $c = \cdot 0001621$.

Pressures.	Ratios.		Absolute temp. of Ethyl Oxide.	
	From observed temperatures.	Smoothed.	Recalculated.	Observed.
millim.				
100	1.0284	1.0283	260.3	260.3
150	1.0298	1.0298	268.5	268.5
200	1.0306	1.0307	274.8	274.8
300	1.0324	1.0325	284.0	284.0
400	1.0335	1.0337	290.95	291.0
500	1.0344	1.0346	296.65	296.7
600	1.0353	1.0355	301.45	301.5
700	1.0361	1.0361	305.7	305.7
800	1.0372	1.0369	309.4	309.3
900	1.0371	1.0375	312.8	312.9
1000	1.0380	1.0380	316.0	316.0
1500	1.0408	1.0402	328.7	328.5
2000	1.0423	1.0420	338.1	338.0
3000	1.0451	1.0446	352.95	352.8
5000	1.0490	1.0484	373.6	373.4

In this series the greatest difference between the observed and recalculated temperatures is only $0^{\circ} \cdot 2$.

5. *Carbon Bisulphide and Ethyl Bromide.* $\frac{\text{CS}_2}{\text{C}_2\text{H}_5\text{Br}}$. Temperatures of carbon bisulphide taken as correct. $c = \cdot 0001185$.

Pressures.	Ratios.		Absolute temperatures of Ethyl Bromide.	
	From observed temperatures.	Smoothed.	Recalculated.	Observed.
millim.				
50	1.0162	1.0172	249.75	250.0
100	1.0186	1.0188	262.8	262.8
150	1.0203	1.0199	271.1	271.0
200	1.0224	1.0206	277.5	277.0
300	1.0223	1.0218	286.9	286.8
400	1.0233	1.0228	294.05	293.9
500	1.0237	1.0234	299.9	299.8
600	1.0244	1.0241	304.8	304.7
700	1.0251	1.0247	309.1	309.0
800	1.0249	1.0251	312.9	313.0
900	1.0259	1.0256	316.4	316.3
1000	1.0263	1.0260	319.7	319.6
1500	1.0274	1.0276	332.7	332.8
2000	1.0280	1.0289	342.4	342.7
3000	1.0299	1.0308	357.7	358.0
5000	1.0327	1.0335	379.0	379.3

6. *Carbon Bisulphide and Ethyl Chloride.* $\frac{\text{CS}_2}{\text{C}_2\text{H}_5\text{Cl}}$. Tem-
peratures of carbon bisulphide taken as correct. $c = \cdot 0001050$.

Pressures.	Ratios.		Absolute temperatures of Ethyl Chloride.	
	From observed temperatures.	Smoothed.	Recalculated.	Observed.
millim.				
150	1·1118	1·1135	248·3	248·7
200	1·1136	1·1142	254·2	254·3
300	1·1152	1·1152	262·9	262·9
400	1·1164	1·1160	269·5	269·4
500	1·1164	1·1166	274·9	274·9
600	1·1174	1·1172	279·4	279·3
700	1·1181	1·1177	283·4	283·3
800	1·1178	1·1181	286·9	287·0
900	1·1186	1·1185	290·1	290·1
1000	1·1194	1·1189	293·15	293·0
1500	1·1210	1·1203	305·2	305·0
2000	1·1209	1·1215	314·1	314·3
3000	1·1224	1·1232	328·25	328·5
5000	1·1253	1·1256	348·0	348·1

7. *Chlorobenzene and Water.* $\frac{\text{H}_2\text{O}}{\text{C}_6\text{H}_5\text{Cl}}$. Temp. of water
taken as correct. $c = \cdot 000598$.

Pressures.	Ratios.		Absolute temperatures of Chlorobenzene.	
	From observed temperatures.	Smoothed.	Recalculated.	Observed.
millim.				
100	1·0579	1·0573	343·3	343·5
150	1·0625	1·0623	353·85	353·9
200	1·0654	1·0660	362·0	361·8
300	1·0711	1·0716	373·9	373·7
400	1·0753	1·0760	383·05	382·8
500	1·0788	1·0791	390·3	390·2
600	1·0819	1·0820	396·55	396·5
700	1·0845	1·0845	402·0	402·0

8. *Bromobenzene and Water.* $\frac{\text{C}_6\text{H}_5\text{Br}}{\text{H}_2\text{O}}$. Temperatures of water taken as correct. $c = \cdot 000609$.

Pressures.	Ratios.		Absolute temperatures of Bromobenzene.	
	From observed temperatures.	Smoothed.	Recalculated.	Observed.
millim.				
50	1·1156	1·1124	348·3	347·3
100	1·1220	1·1206	363·9	364·3
150	1·1255	1·1258	375·0	374·9
200	1·1284	1·1297	383·65	383·2
300	1·1344	1·1353	396·1	395·8
400	1·1388	1·1396	405·7	405·4
500	1·1427	1·1431	413·45	413·3
600	1·1460	1·1460	420·0	420·0
700	1·1489	1·1486	425·8	425·9
800	1·1516	1·1509	430·95	431·2

9. *Aniline and Water.* $\frac{\text{C}_6\text{H}_5\text{NH}_2}{\text{H}_2\text{O}}$. Temperatures of water taken as correct. $c = \cdot 000345$.

Pressures.	Ratios.		Absolute temperatures of Aniline.	
	From observed temperatures.	Smoothed.	Recalculated.	Observed.
millim.				
50	1·2046	1·2048	375·05	375·0
100	1·2100	1·2094	392·7	392·9
150	1·2129	1·2124	403·85	404·0
200	1·2144	1·2145	412·45	412·4
300	1·2174	1·2176	424·8	424·75
400	1·2197	1·2201	434·4	434·2
500	1·2217	1·2221	442·1	441·9
600	1·2237	1·2239	448·55	448·5
700	1·2254	1·2253	454·2	454·25
800	1·2269	1·2266	459·3	459·4

10. *Methyl Salicylate and Water.* $\frac{C_6H_4(OH)COO(CH_3)}{H_2O}$.

Temperatures of water taken as correct. $c = .000459$.

Pressures.	Ratios.		Absolute temperatures of Methyl Salicylate.	
	From observed temperatures.	Smoothed.	Recalculated.	Observed.
millim.				
50	1.3026	1.3008	404.9	405.5
100	1.3067	1.3069	424.35	424.3
150	1.3101	1.3107	436.6	436.4
200	1.3118	1.3136	446.1	445.5
300	1.3170	1.3179	459.8	459.5
400	1.3204	1.3212	470.35	470.05
500	1.3235	1.3239	478.9	478.7
600	1.3261	1.3260	486.0	486.0
700	1.3283	1.3280	492.3	492.4
800	1.3302	1.3298	497.95	498.1

11. *Bromonaphthalene and Water.* $\frac{C_{10}H_7Br}{H_2O}$. Temperatures of water taken as correct. $c = .000612$.

Pressures.	Ratios.		Absolute temperatures of Bromonaphthalene.	
	From observed temperatures.	Smoothed.	Recalculated.	Observed.
millim.				
150	1.4587	1.4592	486.05	485.9
200	1.4623	1.4631	496.9	496.6
300	1.4689	1.4690	512.5	512.5
400	1.4733	1.4732	524.45	524.5
500	1.4766	1.4769	534.2	534.1
600	1.4797	1.4798	542.3	542.3
700	1.4823	1.4821	549.4	549.5
800	1.4846	1.4847	555.95	555.9

12. *Sulphur and Carbon Bisulphide.* $\frac{S}{CS_2}$. Temperatures of carbon bisulphide taken as correct. $c = -.0006845$.

Pressures.	Ratios.		Absolute temperatures of Sulphur.	
	From observed temperatures.	Smoothed.	Recalculated.	Observed.
millim.				
300	2.2783	2.2779	667.9	668.0
400	2.2733	2.2727	683.5	683.7
500	2.2681	2.2685	696.2	696.1
600	2.2650	2.2649	707.0	707.0
700	2.2607	2.2617	716.4	716.1
800	2.2583	2.2588	724.6	724.6
900	2.2558	2.2563	732.2	732.0
1000	2.2530	2.2540	739.3	739.0
1500		2.2444	767.4	
2000	2.2373	2.2372	788.15	788.2
3000	2.2262	2.2262	820.8	820.8

The close agreement between the observed and recalculated temperatures of such a high-boiling substance as sulphur is very striking.

13. *Carbon Bisulphide and Ethylene.* $\frac{CS_2}{C_2H_4}$. Temperatures of carbon bisulphide taken as correct. $c = .000274$.

Pressures.	Ratios.		Absolute temperatures of Ethylene.	
	From observed temperatures.	Smoothed.	Recalculated.	Observed.
millim.				
50	1.8212	1.8208	139.5	139.5
100	1.8242	1.8244	146.8	146.75
150	1.8263	1.8268	151.35	151.4
200	1.8283	1.8287	154.9	154.9
300	1.8325	1.8315	160.1	160.0
400	1.8339	1.8336	164.0	164.0

Here again the agreement is very remarkable, considering the difficulty of measuring very low temperatures accurately.

14. *Water and Oxygen.* $\frac{H_2O}{O}$. Temperatures of water taken as correct. $c = -.0003932$.

Vapour-pressures of oxygen have been determined by Olszewski and by Wroblewski, but their results do not agree

well together. Olszewski measured his temperatures by means of a hydrogen-thermometer; Wroblewski by a thermopile. It was found that the curve showing the relation of temperature to pressure was very fairly regular when constructed from Olszewski's numbers, and the temperatures corresponding to definite pressures were therefore read from this curve. The temperature corresponding to a pressure of 9 millim. is a single observation by Olszewski. The temperatures corresponding to 50, 100, and 150 millim. are taken from Wroblewski's results.

Pressures.	Ratios.		Absolute temperatures of Oxygen.	
	From observed temperatures.	Smoothed.	Recalculated.	Observed.
millim.				
9	4.5967	4.454	63.5	61.5
50	4.0534	4.349	71.6	76.8
100	3.9353	4.2845	75.8	82.5
150	4.0132	4.253	78.6	83.0
300		4.191	83.2	
400		4.182	85.1	
500		4.1405	87.4	
600		4.120	89.0	
700		4.1025	90.4	
800	4.0701	4.090	91.6	92.0
1000	4.0732	4.064	93.7	93.5
1500	4.0536	4.0175	97.9	97.0
2000	4.0520	3.980	101.1	100.0
3000	3.9478	3.924	106.1	105.5
5000	3.8267	3.849	113.2	113.9
10000	3.7044	3.733	124.7	125.7
15000	3.6489	3.656	132.7	133.0
20000	3.6242	3.5975	139.1	138.1

It is hardly to be expected that the measurement of such extremely low temperatures could be very exact, and the difference of 2°C at an absolute temperature of about 60° , is not more than might be expected. It will be seen that the three temperatures given by Wroblewski differ considerably from those calculated from Olszewski's results.

The vapour-pressures of oxygen are given up to the critical point, at which the pressure is 38,600 millim., but those of water have only been determined up to 20,000 millim. The absolute temperatures of oxygen were therefore compared with those of alcohol, in order to find whether the value of c remains constant up to the critical point.

15. *Alcohol and Oxygen.* $\frac{C_2H_5O}{O}$. Temperatures of alcohol taken as correct. $c = -0.00040075$.

Pressures.	Ratios.		Absolute temperatures of Oxygen.	
	From observed temperatures.	Smoothed.	Recalculated.	Observed.
millim.				
300	...	3.940	83.7	
400	...	3.913	85.9	
500	...	3.894	87.6	
600	...	3.879	89.1	
700	...	3.861	90.4	
800	3.8331	3.848	91.6	92.0
1000	3.8332	3.824	93.7	93.5
1500	3.8123	3.780	97.8	97.0
2000	3.7800	3.7485	100.8	100.0
3000	3.7024	3.694	105.7	105.5
5000	3.5848	3.621	112.8	113.9
10000	3.4662	3.514	124.0	125.7
15000	3.4098	3.441	131.8	133.0
20000	3.3824	3.388	137.9	138.1
30000	3.3189	3.300	147.9	147.1
40000	3.2698	3.240	155.6	*154.2

The absolute temperatures of oxygen recalculated from the smoothed ratios with alcohol agree well with those calculated from the ratios with water. It was thought worth while to compare these temperatures with those of sulphur, oxygen being the most and sulphur the least volatile of all the substances examined.

* The last pressure, 40,000 millim., is a little above the critical point, but sufficiently near to allow of the continuation of the vapour-pressure curve without sensible error. This was done instead of calculating the absolute temperature of alcohol corresponding to 38600 millim., the critical pressure of oxygen. It is to be remarked that the last two or three points representing the experimentally determined vapour-pressures of oxygen at the highest temperatures lie a little above the curve which was drawn. If more weight had been given to these points, the agreement between the observed and recalculated numbers would have been much closer. Thus Olszewski gives 154.2 as the absolute temperature at the critical pressure 38600 millim., whereas from the curve this would be the temperature at 40000 millim. If the curve were raised so as to pass through the point representing the observed critical pressure and temperature, the interpolated temperature would become 155.3, or only 0.3 lower than that recalculated from the smoothed ratio.

In any case, the agreement seems to be close enough to show that the equation $R' = R + c(t' - t)$ holds good up to the critical point:

16. Sulphur and Oxygen. $\frac{S}{O}$. Temperatures of Sulphur taken as correct. $c = -.0001652$.

Pressures.	Ratios.			Absolute Temperatures of Oxygen.				
	From observed temperatures.	From mean of recalculated temperatures.	Smoothed.	a. $\frac{H_2O}{O}$.	b. $\frac{C_2H_6}{O}$.	c. Mean of a & b.	From $\frac{S}{O}$.	Observed.
millim.	...	8.005	8.005	83.2	83.7	83.45	83.45	...
300	...	7.9065	7.980	85.1	85.9	85.5	85.7	...
400	...	7.905	7.957	87.4	87.6	87.5	87.5	...
500	...	7.9395	7.9395	89.0	89.1	89.05	89.05	...
600	...	7.921	7.9245	90.4	90.4	90.4	90.4	...
700	...	7.910	7.910	91.6	91.6	91.6	91.6	92.0
800	7.904	7.887	7.8875	93.7	93.7	93.7	93.7	93.5
1000	7.9155	7.847	7.840	97.9	97.8	97.85	97.9	97.0
1500	7.882	7.808	7.8065	101.1	100.8	100.95	101.0	100.0
2000	7.780	7.750	7.7525	106.1	105.7	105.9	105.9	105.5

It will be seen that the temperatures of oxygen recalculated from those of sulphur, when $c = -0001652$, agree remarkably well with the mean of the temperatures recalculated from the ratios with water and alcohol.

17. *Acetic Acid and Water.* $\frac{C_2H_4O_2}{H_2O}$. Temperatures of water taken as correct. For liquid acetic acid $c = 0003186$; for solid acetic acid $c = -000852$.

In the 'Philosophical Transactions,' Part II., 1884, it was shown experimentally by Ramsay and Young that the vapour-pressures of solid acetic acid were lower than those of the liquid acid at the same temperature, and curves were constructed showing the relation of temperature to pressure for both the solid and the liquid acid. The absolute temperatures corresponding to definite pressures were read from these curves. For pressures below 5 millim. the vapour-pressures of water

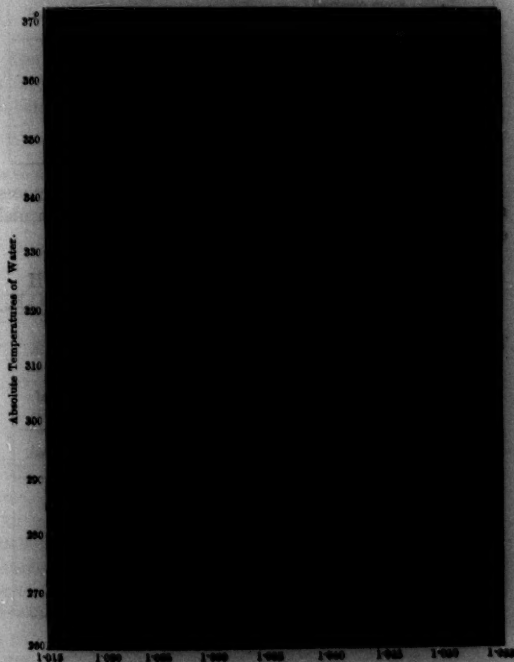
a. Liquid Acetic Acid and Water.				
Pressures.	Ratios.		Absolute temperatures of Acetic Acid.	
	From observed temperatures.	Smoothed.	Recalculated.	Observed.
millim.				
5	1.0175	1.0188	279.35	279.01
6	1.0193	1.0195	282.2	282.14
7	1.0202	1.0202	284.6	284.64
8	1.0212	1.0209	286.8	286.91
9	1.0219	1.0215	288.8	288.88
10	1.0222	1.0220	290.55	290.6
50	1.0311	1.0305	320.8	321.0
100	1.0345	1.0347	336.0	335.9
150	1.0369	1.0375	345.6	345.4
200	1.0392	1.0395	353.0	352.9
300	1.0426	1.0424	363.7	363.75
400	1.0447	1.0447	371.9	371.9
500	1.0455	1.0465	378.5	378.15
600	1.0478	1.0480	384.1	384.0
700	1.0506	1.0495	389.05	389.45
b. Solid Acetic Acid and Water.				
2	1.0423	1.0422	272.9	272.89
3	1.0372	1.0374	277.15	277.10
4	1.0335	1.0335	280.15	280.14
5	1.0306	1.0307	282.6	282.60
6	1.0284	1.0284	284.7	284.67
7	1.0264	1.0261	286.3	286.37
8	1.0244	1.0244	287.8	287.8
9	1.0226	1.0226	289.1	289.08

adopted by Regnault were not employed, the values calculated from his empirical formula representing the relation of temperature to pressure of water between 0° and 100° being probably more correct.

The differences between the recalculated and observed absolute temperatures of solid acetic acid are extremely minute. If the vapour-pressures of water given by Regnault were employed, the value of c would be slightly altered, and in that case the agreement would not be quite so close, the greatest difference amounting to $0^{\circ}\cdot 15$.

The results for both the solid and the liquid acid are shown in the diagram (fig. 2).

Fig. 2.



Dissociating Substances.

Five of these bodies have been examined, three liquid and two solid. In all of them the value of c is constant, while in the case of nitrogen peroxide, when compared with water it is $=0$, or at any rate it is a very small number.

18. *Nitrogen Peroxide and Water.* $\frac{N_2O_4}{H_2O}$. Water taken as correct. $c=0$.

Pressures.	Ratios.		Absolute temperatures of N_2O_4 .	
	From observed temperatures.	Smoothed.	Recalculated.	Observed.
millim.				
100	0.7884	0.790	256.5	256.0
150	0.7899	"	263.15	263.1
200	0.7895	"	268.3	268.1
300	0.7908	"	275.6	275.9
400	0.7904	"	281.25	281.4
500	0.7904	"	285.75	285.9

Considering the nature of the substance, these differences may probably be considered within the limits of experimental error.

19. *Chloral Ethyl-alcoholate and Water.* $\frac{CCl_3CH \begin{cases} OH \\ OC_2H_5 \end{cases}}{H_2O}$.

Temperatures of water taken as correct. $c = -0.0005119$.

Pressures.	Ratios.		Absolute temperatures of Chloral Ethyl-alcoholate.	
	From observed temperatures.	Smoothed.	Recalculated.	Observed.
millim.				
50	1.0729	1.0728	334.0	334.0
100	1.0659	1.0659	346.1	346.1
150	1.0619	1.0615	353.6	353.7
200	1.0545	1.0582	359.4	358.1
300	1.0530	1.0535	367.6	367.4
400	1.0600	1.0499	373.8	373.8
500	1.0470	1.0470	378.7	378.7

20. *Chloral Methyl-alcoholate and Water.* $\text{CCl}_3\text{CH} \begin{cases} \text{OH} \\ \text{OCH}_3. \end{cases}$
 H_2O

Temperatures of water taken as correct. $c = -.0005392$.

Pressures.	Ratios.		Absolute temperatures of Chloral Methyl-alcoholate.	
	From observed temperatures.	Smoothed.	Recalculated.	Observed.
millim.				
50	1.0639	1.0641	331.25	331.2
100	1.0573	1.0570	343.2	343.3
150	1.0532	1.0546	351.3	350.8
200	1.0486	1.0489	356.2	356.1
300	1.0439	1.0439	364.2	364.2
400	1.0402	1.0400	370.2	370.3

The last two substances examined are solid throughout, and, at the higher temperatures at any rate, dissociate completely on being converted into vapour.

21. *Ammonium Chloride and Water.* NH_4Cl
 H_2O . Temperatures of water taken as correct. $c = -.0006667$.

Pressures.	Ratios.		Absolute temperatures of NH_4Cl .	
	From observed temperatures.	Smoothed.	Recalculated.	Observed.
millim.				
50	1.6855	1.6809	523.3	524.7
100	1.6717	1.6720	542.9	542.8
150	1.6650	1.6664	555.1	554.6
200	1.6587	1.6620	564.4	563.3
300	1.6548	1.6560	577.8	577.3
400	1.6514	1.6511	587.8	587.9
500	1.6478	1.6475	595.9	596.0
600	1.6453	1.6441	602.6	603.0

The determinations of the vapour-pressures and temperatures of volatilization of ammonium chloride were attended with great difficulties, and the agreement between the calculated and observed temperatures is as near as could be expected. The volatilizing-point at the atmospheric pressure, calculated from the equation $R' = R + c(t' - t)$, would be $338^{\circ}6\text{ C.}$, which agrees well enough with the temperature found by experiment.

22. *Ammonium Carbamate and Water.* $\text{CO} \begin{Bmatrix} \text{NH}_2 \\ \text{ONH}_2 \end{Bmatrix} \cdot \text{H}_2\text{O}$. Tem-
peratures of water taken as correct. $c = -.000696$.

Pressures.	Ratios.		Absolute temperatures of Ammonium Carbamate.	
	From observed temperatures.	Smoothed.	Recalculated.	Observed.
millim.				
10	0.9529	0.9510	270.4	270.9
50	0.9313	0.9321	290.2	289.9
100	0.9221	0.9250	299.7	299.4
150	0.9159	0.9170	305.45	305.1
200	0.9122	0.9126	309.9	309.8
300	0.9060	0.9061	318.15	316.1
400	0.9017	0.9011	320.8	321.0

It is thought that these twenty-two examples, including twenty-one different bodies—solid, liquid, stable, and dissociable—are sufficient to prove that the equation $R' = R + c(t' - t)$ is applicable to all classes of bodies, and that by the determination of the constant c , which involves only a small amount of experimental work, it becomes possible to calculate the vapour-pressures of any substance, assuming those of water or some other body to be accurately known, within the limits of pressure included in the experimental determinations of the standard substance.

It should be pointed out that those dissociable substances which have been investigated, are either, as in the case of ammonium carbamate and chloride and the compounds of chloral, wholly or almost wholly dissociated on their passage into the gaseous state at the temperatures of observation, or the amount of dissociation is very small, as with nitric peroxide. Until reliable data are obtained, it is perhaps premature to make any complete statement in reference to the behaviour of dissociable bodies.

On the Vapour-Pressures of Mercury.

It will have been observed in the tables in Part I. of this series of papers that mercury appears to differ from all the other substances examined, inasmuch as the values of $\frac{dp}{dt} \cdot t$, when

compared with either water or carbon bisulphide, are not even approximately constant at different pressures, but rise steadily and somewhat rapidly. Again, the absolute temperatures of mercury cannot be calculated from those of water by means of the equation $R' = R + c(t' - t)$, for if Regnault's data are correct, c is not a constant. There appeared, however, to be good reason to believe that Regnault's values were not reliable, and he himself admits that great difficulties were experienced in the investigation of this substance.

It was, therefore, considered desirable to attempt a re-determination of the vapour-pressures of mercury with the greatest possible accuracy at a few temperatures. This has been accomplished, and the results are fully described in *Trans. Chem. Soc.* Jan. 1886. It will be sufficient here to mention that new determinations were made at the boiling-points of methyl salicylate and of sulphur under atmospheric pressure (220° and 450° approximately). Two other determinations at the boiling-point of bromonaphthalene under atmospheric and reduced pressure (280° and 270°), described in the *Trans. Chem. Soc.* xlvii. p. 640, and four determinations of the boiling-point of mercury under atmospheric pressure by Regnault (*Mémoires de l'Académie*, xxi. p. 230), were also made use of in the calculations.

On comparing the absolute temperatures of water with those of mercury at these five pressures, it was found that the value of c in the equation $R' = R + c(t' - t)$ was constant, and that mercury did not form an exception to the law, which has been proved to hold good for all the other substances examined.

By making the absolute temperatures of mercury ordinates, and the ratios of the absolute temperatures of mercury to those of water at the same pressures abscissæ, and drawing a straight line through the five experimentally-determined points, the value of c was ascertained, and vapour-pressures of mercury were calculated for each 5° , from 270° to 520° C. It should be mentioned that slightly different results were obtained when the absolute temperatures of water were made the ordinates; but within the limits of temperature just mentioned, the difference was unimportant.

With the absolute temperatures of mercury as ordinates, the value of c is $\cdot 0004788$.

The values of $\frac{dp}{dt} \cdot t$ for mercury were then determined; for the most part by the method of tangents, but between 150 and 700 mm. from the vapour-pressures calculated for each degree of temperature, and were compared with those of water and of carbon bisulphide. The results are given in the following table:—

Pressure.	$\frac{dp}{dt}$	t	$\frac{dp}{dt} \cdot t$	Water made = 1·000	Carbon bi- sulphide made = 1·000
millim.					
10	0·350	457·3	160	0·865	
50	1·405	508·5	714	0·850	1·057
100	2·506	534·2	1339	0·833	1·046
150	3·54	550·97	1950	0·843	1·061
200	4·47	563·44	2519	0·835	1·061
300	6·25	582·21	3639	0·834	1·061
400	7·85	596·44	4682	0·827	1·055
500	9·40	608·03	5715	0·827	1·055
600	10·91	617·87	6741	0·829	1·060
700	12·34	626·48	7731	0·824	1·057
800	13·75	634·0	8717	0·829	1·059
900	15·01	641·0	9621	0·814	1·037
1000	16·37	647·2	10595	0·818	1·049
1500	22·38	673·1	15064	0·811	1·038
2000	28·33	693·0	19633	0·816	1·064
3000	38·61	723·1	27919	0·804	1·071
5000	56·90	765·6	43562	0·800	1·059

It will be seen that the agreement with carbon bisulphide is extremely close throughout, and that with water the variation is not greater than with some of the other substances examined.

The variations from constancy in the reduced values of $\frac{dp}{dt} \cdot t$ will be considered in Part III.

Note.—Reference has been made to the vapour-pressures of alcohol at high and low temperatures determined by ourselves. The data are contained in a paper presented to the Royal Society in May 1885. The vapour-pressures of dissociable substances are given in a paper read before the Royal Society in November 1885. Neither of these papers is as yet published.

XXXV. *Some Thermodynamical Relations.*—PART III.

By WILLIAM RAMSAY, *Ph.D.*, and SYDNEY YOUNG,
*D.Sc.**

IN the first of this series of papers (*Phil. Mag.* December 1885) two relations between the vapour-pressures of different substances were described:—1. The product of the rate of increase of pressure per unit rise of temperature ($\frac{dp}{dt}$) into the absolute temperature (t) is approximately the same for all bodies at the same pressure, but the differences are real, and are not due to errors of experiment or of calculation. 2. The rate of increase of this value $\frac{dp}{dt} \cdot t$ with rise of pressure is very nearly the same for all bodies. This was shown in the tables by making the product $\frac{dp}{dt} \cdot t$ for one substance equal to unity at each pressure, and reducing the values for other substances at the same pressure in the same ratio. It was pointed out that, at pressures between 150 or 200 millim. and 1500 or 2000 millim., the variations from constancy are very small, but that they are frequently greater at lower and at higher pressures; and it was suggested that these variations might possibly be due either to errors of experiment or of calculation.

In the second paper a relation was proved to exist between the absolute temperature of any two bodies corresponding to equal vapour-pressures, which may be expressed by the equation

$$R' = R + c(t' - t),$$

where R' is the ratio of the absolute temperatures of any two bodies at a given pressure, R the ratio at another given pressure, t' and t are the temperatures of one of the two bodies corresponding to those vapour-pressures, and c is a constant. The value of c may be determined graphically by making the temperatures (absolute or Centigrade) of one of the bodies

* Read December 12, 1885.

(A) ordinates and the ratios of the absolute temperatures of the two bodies (A and B) at pressures corresponding to the temperatures of (A) abscissæ, when the points representing this relation fall in a straight line.

In the case of nearly related bodies such as chlorobenzene and bromobenzene, $c=0$ and $R'=R$ at all pressures, and in those cases the ratios of the products $\frac{dp}{dt} \cdot t$ are also equal at all pressures; and it appeared likely that if the variations from constancy of the reduced values of $\frac{dp}{dt} \cdot t$ for other substances were real, they would be related in some way to the variations from constancy of the ratios of the absolute temperature. This is indeed the case; for if we compare the values of $\frac{dp}{dt} \cdot t$ for any two substances, and make the temperatures of one of them ordinates, and the ratios of the products $\frac{dp}{dt} \cdot t$ abscissæ, we again find that the points representing the relation of the ratios of the product $\frac{dp}{dt} \cdot t$ at definite pressures to the temperatures of one of the bodies corresponding to those pressures, fall in a straight line. We therefore obtain a similar equation,

$$R' = R + c (t' - t),$$

R' and R in this case representing the ratios of the products $\frac{dp}{dt} \cdot t$. The cause of the apparently greater deviation from constancy at low and at high pressures is simply this—that at low pressures a small change of pressure corresponds to a large change of temperature, while at high pressures the intervals of pressure taken are very great, and therefore the intervals of temperature are also great. This is clearly seen by referring to the table of absolute temperatures corresponding to the pressures taken. Taking water as an instance, we have the following numbers:—

P.	<i>t</i> absolute.	Interval of <i>t</i> .	P.	<i>t</i> absolute.	Interval of <i>t</i> .
millim.	°	°	millim.	°	°
10	284.3	27.0	600	368.5	4.2
50	311.3	13.4	700	370.7	3.75
100	324.7	8.4	800	374.45	3.35
150	333.1	6.5	900	377.8	3.05
200	339.6	9.3	1000	380.85	12.35
300	348.9	7.1	1500	393.2	9.3
400	356.0	5.7	2000	402.5	14.0
500	361.7	4.8	3000	416.5	19.35
600	366.5		5000	435.85	

The determinations of the products $\frac{dp}{dt} \cdot t$ are, however, necessarily less accurate than those of the temperatures, especially at low pressures, for the sources of error are more numerous and have greater influences on the results; yet the values calculated directly from the vapour-pressure curves generally agree very well with those recalculated from the equation

$$R' = R + c(t' - t),$$

as will be seen in the following tables. With the exception of chlorobenzene and bromobenzene, the comparison is only made for those substances which have been investigated through a large range of pressure, for otherwise it is impossible to determine the value of *c* with sufficient certainty.

1. *Water and Carbon Bisulphide.* Ratios of the products $\frac{dp}{dt} \cdot t$ for CS_2 to those of water at the same pressures taken as abscissæ, and the temperatures of water corresponding to those pressures taken as ordinates. $c = -0.000399$.

Pressures.	Reduced ratios of $\frac{dp}{dt} \cdot t$, calc. from curves.	Reduced ratios of $\frac{dp}{dt} \cdot t$, recalcd. from equation.	Products $\frac{dp}{dt} \cdot t$ for CS_2 , calc. from curves.	$\frac{dp}{dt} \cdot t$ for CS_2 , recalcd. from values for water and recalcd. ratios.	Δ per cent.
millim.					
50	804	8042	675.7	675.9	+0.03
100	796	7989	1280.0	1284.0	+0.31
150	794	7955	1839	1842	+0.16
200	787	7929	2373	2391	+0.75
300	787	7892	3430	3442	+0.35
400	784	7864	4436	4451	+0.34
500	784	7841	5417	5417	0.0
600	782	7822	6361	6364	+0.05
700	780	7805	7317	7320	+0.04
800	783	7790	8235	8197	-0.46
900	785	7777	9281	9196	-0.92
1000	780	7764	10102	10054	-0.48
1500	781	7715	14507	14325	-1.27
2000	767	7678	18460	18481	+0.11
3000	750	7622	26056	26476	+1.59
5000	755	7545	41128	41106	-0.05

There are only two cases in which the difference between the calculated and recalculated values amounts to 1 per cent., and these two cases occur close together and are in opposite directions. Generally the differences are much smaller than 1 per cent., while the differences between the highest and lowest of the reduced values of $\frac{dp}{dt} \cdot t$ (calculated or recalculated) exceeds 6 per cent.

2. *Bromo-benzene and Chloro-benzene.* The values for these substances appeared to be identical. They were therefore taken together and compared both with carbon bisulphide and with water, to find whether the recalculated values would be the same in the two comparisons. This is shown by the table opposite to be the case.

a. Values of $\frac{dp}{dt} \cdot t$ for CS_2 taken as unity at each pressure. Temperatures of CS_2 as ordinates. $c = -0.0007626$.

b. Values of $\frac{dp}{dt} \cdot t$ for H_2O taken as unity. Temperatures of H_2O as ordinates. $c = -0.0009850$.

Pres- sures.	$\text{CS}_2=1$.				$\text{H}_2\text{O}=1$.				$\frac{dp}{dt} \cdot t$ for chloro- and bromo-benzene. Recalculated.	
	Mean of re- duced ratios of $\frac{dp}{dt} \cdot t$ for chloro- and bromo- benzene. Calculated.	Reduced ratios of $\frac{dp}{dt} \cdot t$ for chloro- and bromo-benzene. Recalculated.	Mean of re- duced ratios of $\frac{dp}{dt} \cdot t$ for chloro- and bromo- benzene. Calculated.	Reduced ratios of $\frac{dp}{dt} \cdot t$ for chloro- and bromo-benzene. Recalculated.	$\frac{dp}{dt} \cdot t$ for chloro- benzene. Calculated.	$\frac{dp}{dt} \cdot t$ for bromo- benzene. Calculated.	Mean values of $\frac{dp}{dt} \cdot t$ for chloro- and bromo- benzene.	$\text{CS}_2=1$.	$\text{H}_2\text{O}=1$.	
										millim.
50	1.095	1.0974	0.880	0.8785	...	740	740	741.5	738.4	
100	1.086	1.0870	0.8645	0.8633	1384	1395	1390	1391	1391	
150	1.068	1.0803	0.848	0.8570	1982	1946	1964	1987	1984	
200	1.071	1.0732	0.8425	0.8506	2568	2525	2542	2551	2565	
300	1.068	1.0675	0.8405	0.8415	3670	3657	3664	3662	3670	
400	1.0625	1.0618	0.833	0.8345	4724	4703	4714	4710	4723	
500	1.067	1.0671	0.829	0.8289	5728	5724	5726	5726	5726	
600	1.055	1.0531	0.8245	0.8241	6740	6678	6709	6699	6705	
700	1.050	1.0496	0.8195	0.8200	7678	7688	7683	7680	7691	
800	1.045	1.0465	0.818	0.8163	...	8602	8602	8618	8589	

3. *Carbon Tetrachloride and Water.* Values of $\frac{dp}{dt} \cdot t$ for H_2O taken as unity at each pressure. Temperatures of water as ordinates. $c = -0.0006945$.

Pressures.	Reduced ratios of $\frac{dp}{dt} \cdot t$.		$\frac{dp}{dt} \cdot t$ for CCl_4 .	
	Calculated.	Recalculated.	Calculated.	Recalculated.
millim.				
50	·859	·8455	722	710·6
100	·830	·8362	1333	1344
150	·827	·8304	1915	1922
200	·832	·8258	2510	2491
300	·811	·8194	3537	3573
400	·809	·8145	4580	4610
500	·823	·8105	5684	5599
600	·819	·8072	6669	6567
700	·794	·8042	7446	7543
800	·799	·8016	8403	8435
900	·789	·7993	9330	9452
1000	·785	·7972	10170	10323
1500	·779	·7886	14470	14642
2000	·780	·7822	18783	18828
3000	·765	·7724	26564	26830
5000	·759	·7590	41344	41351

4. *Ether and Water.* Values of $\frac{dp}{dt} \cdot t$ for H_2O taken as unity at each pressure. Temperatures of water as ordinates. $c = -0.0003194$.

Pressures.	Reduced ratios of $\frac{dp}{dt} \cdot t$.		$\frac{dp}{dt} \cdot t$ for $(C_2H_5)_2O$.	
	Calculated.	Recalculated.	Calculated.	Recalculated.
millim.				
100	·811	·8905	1304	1335
150	·833	·8278	1928	1916
200	·839	·8257	2531	2490
300	·825	·8228	3598	3588
400	·840	·8205	4752	4644
500	·812	·8187	5612	5655
600	·826	·8171	6723	6648
700	·818	·8158	7673	7651
800	·811	·8146	8530	8571
900	·808	·8135	9559	9620
1000	·812	·8126	10513	10523
1500	·798	·8086	14819	15013
2000	·822	·8057	19787	19398
3000	·798	·8012	27730	27830
5000	·795	·7950	43314	43313

5. *Chloroform and Water.* Values of $\frac{dp}{dt} \cdot t$ for water taken as unity at each pressure. Temperatures of water as ordinates. $c = -0.0006701$.

Pressures.	Reduced ratios of $\frac{dp}{dt} \cdot t$.		$\frac{dp}{dt} \cdot t$ for COI_3H .	
	Calculated.	Recalculated.	Calculated.	Recalculated.
millim.				
200	854	8545	2575	2577
300	836	8483	3646	3699
400	843	8436	4772	4775
500	844	8397	5829	5801
600	842	8365	6850	6806
700	838	8337	7862	7819
800	838	8311	8812	8745
900	821	8289	9709	9802
1000	838	8269	10846	10708
1500	818	8186	15191	15199
2000	805	8123	19364	19552
3000	801	8030	27813	27893
5000	777	7900	42320	43041

6. *Ethyl Bromide and Ethyl Chloride with Water.* Values of $\frac{dp}{dt} \cdot t$ for water taken as unity at each pressure, and compared with mean values of $\frac{dp}{dt} \cdot t$ for ethyl bromide and ethyl chloride. Temperatures of water as ordinates. $c = -0.0006335$.

Pressures.	Mean of reduced ratios of $\frac{dp}{dt} \cdot t$.		Mean values of $\frac{dp}{dt} \cdot t$ for $\text{C}_2\text{H}_5\text{Br}$ and $\text{C}_2\text{H}_5\text{Cl}$.	
	Calculated.	Recalculated.	Calculated.	Recalculated.
millim.				
50	851	8449	715	710.1
100	849	8364	1364	1344
150	829	8311	1919	1924
200	823	8271	2483	2495
300	8085	8211	3525	3581
400	8045	8166	4555	4622
500	8085	8130	5587	5616
600	817	8099	6647	6589
700	8085	8073	7580	7572
800	8045	8049	8463	8469
900	7955	8028	9407	9493
1000	798	8008	10330	10370
1500	793	7930	14721	14733
2000	790	7871	18994	18946
3000	7835	7783	27207	27035
5000	7615	7660	41504	41733

7. *Mercury and Water.* From determinations of vapour-pressures of mercury by Ramsay and Young*. If Regnault's values are correct, the equation $R' = R + c(t' - t)$ does not hold good. Values of $\frac{dp}{dt} \cdot t$ for water taken as unity at each pressure. Temperatures of water as ordinates. $c = -0.0004236$.

Pressures.	Reduced ratios of $\frac{dp}{dt} \cdot t$.		$\frac{dp}{dt} \cdot t$ for Mercury.	
	Calculated.	Recalculated.	Calculated.	Recalculated.
millim.				
10	866	8600	160	158.9
50	850	8486	714	713.2
100	833	8429	1339	1355
150	843	8393	1950	1943
200	835	8366	2519	2523
300	834	8326	3639	3631
400	827	8296	4682	4696
500	827	8272	5715	5714
600	829	8252	6741	6714
700	824	8234	7731	7723
800	829	8218	8717	8647
900	814	8204	9621	9701
1000	818	8191	10595	10607
1500	811	8139	15064	15112
2000	816	8099	19633	19494
3000	804	8040	27919	27928
5000	800	7958	43562	43356

8. *Mercury and Carbon Bisulphide.* Employing the values of $\frac{dp}{dt} \cdot t$ for mercury and carbon bisulphide, both recalculated from their ratios with those of water, these values bear an almost constant ratio to each other at all pressures, or $c=0$. The calculated ratios vary only between 1.0547 and 1.0552.

XXXVI. *The Winding of Voltmeters.* By Professors W. E. AYRTON, F.R.S., and JOHN PERRY, F.R.S.†

I. THE use of voltmeters for measuring the potential-difference existing between the mains in electric light

* The data for these determinations are to be found in the 'Journal of the Chemical Society,' January 1886.

† Read June 13, 1885.

circuits, and for enabling us to maintain this potential-difference constant, is so wide in its possible application, that it is unnecessary to dwell on the importance of voltmeters giving accurate indications. Indeed, the sensibility of an incandescent lamp is so great (the light, as we have shown in a previous communication to this Society*, being in many cases proportional to the cube of the potential-difference minus a constant), that it is necessary that voltmeters should record with great accuracy the potential-difference, as well as quickly indicate any changes that from time to time take place in this potential-difference.

It is therefore desirable to consider what are the most important errors in commercial voltmeters as at present constructed, and how these errors may be practically diminished or altogether removed. They are:—

1. An error arising from the sensibility of the voltmeter varying with its resistance, and therefore with its temperature. This change of temperature is due partly to the variation of the temperature of the room and partly to the coils of the instrument becoming heated by the passage of the current through them.
2. An error arising from the sensibility of the instrument being temporarily varied by external magnetic disturbance.
3. An error arising from certain types of voltmeters altering in sensibility so that their constants vary from time to time.
4. An error arising from the volt standards employed by different makers of voltmeters differing several per cent. from one another.

The variation in the resistance of a voltmeter produced by changes of temperature of the room can of course be made very small by winding the instrument with German-silver wire, or better with platinum-silver, or, best of all, with wire made of platinoid, or it may be made absolutely nought by use of an outside resistance of carbon; or, lastly, if the voltmeter be simply wound with copper wire, this error, although large, can be accurately allowed for if the temperature of the voltmeter be known. But the error arising from the heating

* "The most Economical Potential-Difference to employ with Incandescent Lamps," Proc. Phys. Soc. vol. vii. p. 40 [Phil. Mag. [5] vol. xix. p. 304 (April 1885)].

of the voltmeter by the passage of the current through it is much more difficult to allow for, as this rise of temperature of the instrument will be unknown. Hence it is important to reduce this error to a minimum, and it does not at all follow that the employment of German-silver wire or even platinoid wire will effect this result; indeed we have already given, in a previous communication made to this Society*, general reasons showing that German silver is undesirable to be used in the winding of voltmeters. When, however, the diameter of the coil of the instrument exceeds a certain limit, it would appear that German-silver wire might be better for the outer coils: hence the following investigation has been made for the purpose of ascertaining whether in an instrument of the size of our Solenoid, or Long-Range Voltmeter, any portion of the coil should be wound with German-silver wire, and if so, how large a portion, also what should be the law of variation of sectional area of the copper and German-silver wire with the diameter of the convolution in which it is wound, so that the heating-error produced by the passage of the current through the voltmeter shall be a minimum.

Although the dimensions of the instrument adopted in the following investigation are those of this special form of voltmeter, the mode of considering the problem will apply equally well to any other galvanometer used as a voltmeter, or to the fine wire coil of an Ohm-meter, Powermeter, and Ergmeter, or indeed in any case where it is desired to produce a given magnetic effect with a minimum error due to heating. It is only necessary to consider the case of a cylindrical coil of internal radius r_0 and external r_1 and length l , as it is easy to extend our reasoning to a coil of any other shape and dimensions.

II. At a place in the coil at a distance r from the axis let the cross section of the wire be x , let ρ be the specific resistance of the material, and γ the rate at which ρ increases with temperature; x , ρ , and γ being functions of r . For simplicity of calculation we shall neglect the volume of the insulating

* "Direct-reading Electro-Measuring Instruments, and a Non-Sparking Key," Proc. Phys. Soc. vol. vi. p. 69 [Phil. Mag. [5] vol. xvii. p. 304 (April 1884)].

material. Let one spire of radius r produce a magnetic effect of the amount

$$KC r^d, \quad \dots \dots \dots (1)$$

C being the current, and d having in almost all instruments a value which lies between the limits -1 and 0 . Also let

$$x = x_0 r^a \quad \dots \dots \dots (2)$$

$$\rho = \rho_0 r^b \quad \dots \dots \dots (3)$$

$$\rho\gamma = \rho_0 \gamma_0 r^c \quad \dots \dots \dots (4)$$

In a layer of wire of thickness δr there are $\frac{l}{x} \delta r$ spires of wire, and the resistance of the layer is

$$\frac{2\pi r \rho l}{x^2} \delta r, \quad \text{or} \quad \frac{2\pi r \rho_0 \gamma_0^d l}{x_0^2 r^{2a}} \delta r, \quad \text{or} \quad \frac{2\pi \rho_0 l r^{1+b-2a}}{x_0^2} \delta r;$$

so that the whole resistance of the coil is evidently

$$R = \frac{2\pi \rho_0 l}{m x_0^2} (r_1^m - r_0^m) \text{ if } m = 2 + b - 2a. \quad \dots (5)$$

The magnetic effect of the spires in the thickness δr is

$$KC r^d l \frac{\delta r}{x} \quad \text{or} \quad \frac{KC l}{x_0} r^{d-a} \delta r,$$

so that the whole magnetic effect of the coil is

$$M = \frac{KC l}{x_0 n} (r_1^n - r_0^n) \text{ if } n = d - a + 1. \quad \dots (6)$$

We shall now assume that we can, by introducing thin strips of copper among the windings, give the same ease of cooling to unit volume of the wire everywhere; that is, after the current has flowed for some time, that the increase of temperature at any point is proportional to the rate at which current heat is produced per unit volume. Now

$$C^2 \frac{2\pi \rho_0}{x_0^2} r^{1+b-2a} l \delta r \quad \text{or} \quad C^2 \frac{\rho_0}{x_0^2} r^{b-2a} \cdot 2\pi r l \delta r$$

is the rate at which heat is produced in a layer of thickness δr .

This rate is $\frac{\rho_0}{x_0^2} r^{b-2a}$ times the volume of the layer. Therefore the rate at which heat is produced *per unit volume* is $C^2 \frac{\rho_0}{x_0^2} r^{b-2a}$: therefore, by hypothesis, the increase of temperature is proportional to $C^2 \frac{\rho_0}{x_0^2} r^{b-2a}$.

The increase of resistance of this layer is the resistance of the layer multiplied by γ times the increase of temperature, and is therefore proportional to

$$\frac{2\pi\rho_0 l r^{1+b-2a}}{x_0^2} \delta r \cdot \gamma \cdot \frac{C^2 \rho_0}{x_0^2} r^{b-2a},$$

or to

$$\gamma \frac{C^2 \rho_0^2}{x_0^4} r^{1+2b-4a},$$

where $\gamma = \gamma_0 r^{c-b}$, and the whole increase of resistance is proportional to

$$\int_{r_0}^{r_1} \gamma_0 C^2 \frac{\rho_0^2}{x_0^4} r^{1+b+c-4a} dr,$$

or

$$\gamma_0 \frac{C^2 \rho_0^2}{p x_0^4} (r_1^p - r_0^p), \text{ if } p = 2 + b - 4a + c; \dots (7)$$

so that the fractional change of resistance is proportional to

$$\frac{\gamma_0 C^2 \rho_0^2}{R x_0^4 p} (r_1^p - r_0^p). \dots (8)$$

Now regarding M as constant, and substituting for C and R from (6) (5) and (8) we observe that the fractional change of resistance due to heating is proportional to

$$\gamma_0 \rho_0 \cdot \frac{n^2}{(r_1^n - r_0^n)^2} \cdot \frac{m}{r_1^m - r_0^m} \cdot \frac{r_1^p - r_0^p}{p} \dots (9)$$

and it is required to find the values of a , b , and c for which this expression is a minimum.

III. It may be seen that the error (9) is less as $\gamma_0 \rho_0$ becomes less, therefore $\gamma_0 \rho_0$ should be made as small as possible so that *when wires of two metals, as for example copper and German silver, are employed, the wire for which γp has the least value, that is the copper, should be used for the inner winding.* It is, however, when we consider the waste of electric energy in the coil, that the great objection to the use of German silver for the internal winding becomes still more striking. For the rate of production of heat in the coil is proportional to $C^2 R$, which, from (6) and (5), equals

$$\frac{x_0^2 n^2 M^2}{K^2 l^2 (r_1^n - r_0^n)} \times \frac{2\pi \rho_0 l}{m x_0} (r_1^m - r_0^m).$$

Hence to produce the same magnetic effect M in a coil of given length l , the rate of production of heat is proportional to

$$\frac{n^2}{(r_1^n - r_0^n)^2} \times \frac{r_1^m - r_0^m}{m} \rho_0. \quad (10)$$

Consequently when a given angular deflection is produced, the rate of wasting electrical energy in heat will be thirteen times as great if German-silver wire be used as it will be if copper is used, no matter what be the gauge of wire employed, that is, whether the voltmeter be wound to measure a maximum potential-difference of 50 or of 500 volts.

Eliminating r from (3) and (4) it follows that

$$\left(\frac{\gamma\rho}{\gamma_0\rho_0}\right)^{\frac{1}{c}} = \left(\frac{\rho}{\rho_0}\right)^{\frac{1}{b}}, \quad (11)$$

where b may have any arbitrary value. Hence if there is a regular change possible in the value of $\gamma\rho$ such as given in (4), we must have

$$c = Pb,$$

where P is some constant. Equation (11), therefore, becomes

$$\frac{\gamma\rho}{\gamma_0\rho_0} = \left(\frac{\rho}{\rho_0}\right)^P. \quad (12)$$

If, therefore, our wire in the inside of the coil is of copper and on the outside of German silver,

$$\gamma\rho = \frac{13}{9} \gamma_0\rho_0,$$

or

$$\rho = 13 \rho_0.$$

Hence from (11)

$$\frac{13}{9} = 13^P,$$

or

$$P = 0.143,$$

and

$$c = 0.143 b. \quad (13)$$

It is certain that generally the relationship between γ and ρ cannot be of the simple character here assumed; but inasmuch as our results are only applied in cases where the windings are either of German silver or of copper, or of the two

together, and in these cases the expressions used are quite correct, there is no practical objection to the use of such a law. The law is very nearly the same as the assumption that a wire of any material, lying electrically between German silver and copper, is made up of wires of German silver and copper.

Since from (7),

$$p = 2 + b - 4a + c,$$

it follows, using (13), that

$$p = 2 + 1.143b - 4a, \dots \dots \dots (14)$$

and the values of m and n are given respectively in terms of a , b , and d in (5) and (6), and are

$$m = 2 + b - 2a \dots \dots \dots (14)$$

$$n = d - a + 1. \dots \dots \dots (14)$$

In our Solenoid Voltmeter, r_0 equals 1, r_1 equals 8, and d equals -1 ; and since the greatest value of ρ is $13\rho_0$, it follows that r^b cannot be greater than 13, but b is a constant, and the greatest value of r is 8, therefore b must be less than 2. From (14), since b is always small, we see, after a has reached a certain large value, that p , m , and n increase numerically as a increases, and are positive when a is negative, and *vice versa*; also that, finally, p equals $2m$ or $4n$.

Substituting the values for r_0 and r_1 , the error (9) may be written

$$\frac{\gamma_0 \rho_0 \left(\frac{n}{8^n - 1} \right)^2 \left(\frac{m}{8^m - 1} \right)}{\frac{p}{8^p - 1}},$$

and it may be seen by inspection that each of the factors of which it is composed is always positive, whether m , n , and p be positive or negative. Therefore the error is always positive. For large values of m , n , and p , the error may be written

$$\frac{\left(\frac{n}{8^n - 1} \right) \left(\frac{2n}{8^{2n} - 1} \right)}{\frac{4n}{8^{4n} - 1}},$$

which equals

$$\frac{\left(\frac{n}{8^n-1}\right)^2}{\frac{2}{8^{2n}+1}},$$

or

$$\frac{n^2(2n+1)}{8^{2n}+1-2 \times 8^n}.$$

This expression is always greater than n^2 whether n be positive or negative, and is therefore very great since n is great. Consequently for large positive or negative values of m , n , and p , the error (9) becomes very great, and therefore it can have no maximum. The values of a and b , therefore, found from equating to nought its differential coefficients with respect to a and to b , will give a minimum value of this error. And the error will be a minimum when its logarithm is a minimum. Now taking its logarithm, putting r_0 equal to 1, and r_1 equal to 8, differentiating with regard to a and b respectively, and equating to zero, we arrive at the equations

$$\phi(n) - 0.857 \phi(p) = 0, \quad \dots \dots \dots (15)$$

$$\phi(m) - 1.143 \phi(p) = 0, \quad \dots \dots \dots (16)$$

where

$$\phi(a) = \frac{\log_e 8}{1-8^{-a}} - \frac{1}{a}. \quad \dots \dots (17)$$

To assist in the solution of these simultaneous equations we calculated the following Table:—

TABLE I.

a .	$\phi(a)$.
— 3	0.3293
— 2	0.4670
— 1	0.7029
0	1.0397
1	1.3764
2	1.6124
3	1.7502
4.5	1.8474
10	1.9794
100	2.0694

And in order to solve these equations we used the following artifice:—A curve was first plotted on squared paper with $\phi(a)$ as ordinate and a as abscissa, and by means of this curve for any value of $\phi(a)$ given we could at once find a approximately.

Table II. was then arrived at in the following way:— value of p was chosen, and $\phi(p)$ written down. Then since $0.857\phi(p)$ is equal to $\phi(n)$ by (15), n was known from this by the curve for $\phi(a)$; m was next found in the same way from (16). Thus columns 1 and 4 in Table II. contain values of p and n which satisfy (15), and which were obtained in this way, while in columns 1 and 6 we have values of p and m which satisfy (16).

TABLE II.

p .	$\phi(p)$.	$0.857\phi(p)$, or $\phi(n)$.	n .	$1.143\phi(p)$, or $\phi(m)$.	m .
2	1.6124	1.381	1.03	1.842	4.35
1	1.3764	1.18	0.39	1.572	1.78
0	1.0397	0.8913	-0.425	1.189	0.425
-1	0.70295	0.6026	-1.370	0.8035	-0.69
-2	0.467	0.4002	-2.38	0.5337	-1.69

Now first using the pair of columns 1 and 4, we chose corresponding values of p and n , say 2 and 1.03; then, since from (14)

$$2 = 2 + 1.143b - 4a,$$

and

$$1.03 = d - a + 1,$$

we could calculate a pair of values for a and b which satisfy (14) and (15), since d was assumed to be known. Next taking another pair of values of p and n from Table II., another pair of values of a and b was calculated, and so on. Plotting these values of a and b as coordinates of points on squared paper we obtained the curve sss , fig. 1*, the coordinates of every point of which are the values of a and b that satisfy (15). In the same way we drew the curve ttt for (16), and the coordinates of the point of intersection Z are the values of a and b which satisfy (15) and (16), and therefore make the error (9) a minimum.

Thus for example, taking $d = -1$, as seems to be the case in our solenoid instrument, we find that, with some approximation to accuracy,

$$a = 0.19 \text{ and } b = -0.82$$

* This and the two succeeding figures are only about one twelfth of the size each way of the figures actually employed in the calculations.

Fig. 1.



for minimum error. But if we have used our material of least resistance for the inner winding, b cannot in practice be less than 0, for ρ cannot be less than ρ_0 , and since r is never less than 1, ρ would be some fraction of ρ_0 if b were less than 0. Hence in practice we find that b must equal 0, that is, *a less heating-error is produced when all the wire is of copper than when we use German-silver windings on the outside of the coil.*

IV. Making b equal to nought as the least value of b attainable, we can now find the value of a to make the error a minimum. To do this we must put b equal to nought, take d equal to -1 in (9), differentiate with respect to a , and equate the result to 0.

$$\text{Thus } \phi(m) + \phi(n) - 2\phi(p) = 0, \quad (18)$$

where $\phi(a)$ has the signification given to it in (17).

$$\text{Since } b = 0$$

$$\text{and } d = -1,$$

it follows from (14) that

$$\left. \begin{aligned} m &= 2 - 2a \\ n &= -a \\ p &= 2 - 4a \end{aligned} \right\} \quad (19)$$

therefore, from (17)

$$\phi(2 - 2a) + \phi(-a) - 2\phi(2 - 4a) = 0.$$

This equation cannot be solved generally, but by giving various values to a we can find which of these values makes the expression on the left-hand side most nearly equal to nought, and to ascertain the value of the function for each of the values of a we can conveniently use the curve connecting $\phi(a)$ and a previously employed.

For $a=0.5$ we have

$$\phi(1) + \phi(-0.5) - 2\phi(0) = 1.375 + 0.865 - 2 \times 1.04 \\ = +0.16;$$

for $a=0.4$ we have

$$\phi(1.2) + \phi(-0.4) - 2\phi(0.4) = 1.43 + 0.9 - 2 \times 1.182 \\ = -0.034;$$

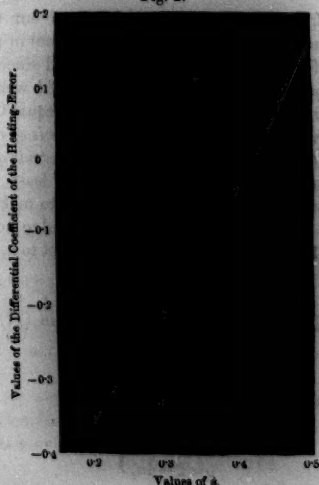
for $a=0.3$ we have

$$\phi(1.4) + \phi(-0.3) - 2\phi(0.8) = 1.48 + 0.936 - 2 \times 1.311 \\ = -0.206;$$

for $a=0.2$ we have

$$\phi(1.6) + \phi(-0.2) - 2\phi(1.2) = 1.53 + 0.97 - 2 \times 1.43 \\ = -0.36.$$

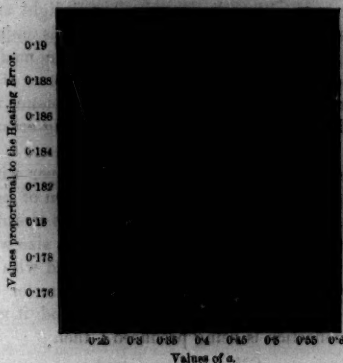
Fig. 2.



Plotting these points, we find (fig. 2) that the curve for

the differential coefficient cuts the axis along which a is measured at a point 0, for which a has the value 0.42; or, in other words, *a equal to 0.42 makes the differential coefficient equal to nought, and hence is the value of a that makes the heating-error a minimum.* As a check on this we have drawn the curve for the values of the heating-error for different values of a (fig. 3), and from this also it is seen that the

Fig. 3.



heating-error has a minimum when a equals 0.42. Hence it follows that the law of increase of the sectional area of the wire with the radius of the convolutions should in our Solenoid Voltmeter follow the law

$$x = x_0 r^{0.42},$$

in order that the error due to heating may be a minimum.

Fig. 3 not only gives the value of a that makes the heating-error a minimum, but it shows by how much the heating-error is increased if a has a value greater or less than 0.42. For example, if a equals 0, that is to say if the voltmeter be all wound with wire of the same gauge, then, continuing the curve backwards, we find that the heating-error is increased by about one third.

V. One plan of diminishing the heating-error with voltmeters, and one that we have frequently employed, is to place

in the same circuit with the instrument a coil wound with German-silver wire, and having a resistance far greater than that of the voltmeter itself. By using wire of large gauge, wound on a bobbin of large volume, and with as efficient an arrangement for cooling as can be conveniently employed, the heating-error can be rendered extremely small. Such a bobbin, however, is both costly and bulky, and the employment of such an outside resistance-coil wastes energy, and can only be used with voltmeters in electric light and power installations where such a waste of energy is unimportant. The following investigation has therefore been made for the purpose of determining what must be the volume of this outside resistance-coil, and what must be its resistance relatively to that of the voltmeter-coil, so that the heating-error may not exceed any given value.

Let R be the resistance of the coil of copper wire wound on a voltmeter, let H be the rate of production of heat when the maximum deflection is obtained; then, as already shown, H will be constant for a voltmeter of a given size wound according to a definite law, whether it be wound with fine wire for a 500-volt instrument or with thicker wire for a 50-volt one. Let C be the current passing round it when this maximum deflection is being produced; then

$$C^2 R = H, \text{ a constant.}$$

Let the increase of resistance produced by the passage of the current for some time be r ; then $\frac{r}{R}$ is proportional to the product of γ , the coefficient of increase of the resistance of copper per degree of temperature, into H , since, as already explained, the voltmeter is so wound that the rate of production of heat is proportional to the increase of temperature. Hence

$$\frac{R}{r} = \text{a constant, A say.}$$

Now let there be a resistance-coil of German-silver wire of uniform cross section, of volume V in series with the instrument, and let us suppose that for any coils thus added the rise of temperature is proportional to the energy wasted per second per unit volume, that is to say proportional to the rate of production of heat per unit volume.

Let R_1 be the resistance of this added coil, then $\frac{C^2 R_1}{V}$ is proportional to its rise of temperature; hence, if r_1 be the increase of resistance and γ_1 the coefficient of increase of resistance of German silver per degree,

$$\frac{r_1}{R_1} \propto \gamma_1 \frac{C^2 R_1}{V},$$

or

$$r_1 = B \frac{C^2 R_1^2}{V} \text{ say,}$$

where B is a constant. Hence, if e is the error due to the proportional change of resistance,

$$e = \frac{r + r_1}{R + R_1},$$

$$= \frac{AR + \frac{BC^2 R_1^2}{V}}{R + R_1}.$$

Let

$$\frac{R_1}{R} = y,$$

then

$$e = \frac{A + \frac{B}{V} y^2}{1 + y} \dots \dots \dots (20)$$

To determine the value of the constants A and B we may proceed as follows:—Wind a voltmeter according to the law which is to be adopted with any or no resistance-coil outside; find the potential-differences P_1 and P_2 required to produce the maximum reading of the instrument,

1st, immediately on making connection;

2nd, after connection has been made for some time;
then

$$e = \frac{P_2 - P_1}{P_1}.$$

Let e be experimentally determined in two cases,—one where there is no resistance-coil outside, and the other where there is such an auxiliary resistance-coil, obtaining values e_1 and e_2 respectively. Measure y , that is the ratio of R_1 to R , and also V . Then from the equations

$$e_1 = A,$$

$$e_2 = \frac{A + \frac{B}{V} y^2}{1 + y},$$

the values of the constants A and B can be determined; and when determined, the error arising with the employment of an outside resistance-coil of any resistance and of any volume can be calculated from (20).

It is obvious that as V becomes greater the error becomes less and less, but there is a practical inconvenience in largely adding to the weight of the instrument.

VI. *To determine the best resistance to give to the outside coil of given volume.*—When V is fixed, e is a maximum or minimum, when

$$\frac{de}{dy} = 0.$$

Now

$$\frac{de}{dy} = \frac{\frac{B}{V}y^2 + 2\frac{B}{V}y - A}{(1+y)^2},$$

and

$$\frac{d^2e}{dy^2} = \frac{2\left(\frac{B}{V} + A\right)}{(1+y)^3},$$

y must be positive, therefore $(1+y)^3$ is positive, also A and B are both positive, therefore $\frac{d^2e}{dy^2}$ is positive, and therefore e is a minimum, when

$$\frac{de}{dy} = 0,$$

that is, when

$$\frac{B}{V}y^2 + 2\frac{B}{V}y - A = 0,$$

or

$$y = -1 \pm \sqrt{1 + \frac{AV}{B}};$$

but y cannot be negative, therefore only the positive sign can be used, or

$$y = \sqrt{1 + \frac{AV}{B}} - 1. \quad (21)$$

makes e a minimum.

Substituting this value of y in (20), we find that

$$e = 2 \frac{A \frac{B}{V} - \sqrt{1 + \frac{AV}{B}}}{\sqrt{1 + \frac{AV}{B}}} \quad (22)$$

VII. *To determine the best volume, and the best resistance to give to the outside coil for a given permissible error.*—From the preceding it is easy to see that if the maximum allowable value of e is given, the minimum value of the volume that can be employed is given from (22), and for this value of the volume, the best value of y , that is the best ratio of the resistance of the outside coil to that of the magnetizing coil of the voltmeter, can be calculated from (21).

VIII. *To determine the sectional area to be given to the inner wire of a voltmeter in terms of the maximum potential difference so as to make the heating-error a minimum.*—By experiment ascertain the potential difference that is necessary to produce the maximum deflection with any voltmeter of the shape and size to be employed and wound according to the law

$$x = x_0 r^a,$$

where a has the value that makes the heating-error a minimum, and which value will depend on the shape and size of the instrument, and may be calculated for any special type of voltmeter in the same way as it has been calculated in what precedes for the Long-Range Magnifying Spring-instrument. Then if it is desired to wind another instrument, of the same volume and shape, so that it will have its maximum deflection given by P_2 volts, it follows, since the heating will, as already proved, be the same in the two instruments for their maximum deflections, that their resistances must be proportional to the squares of P_1 and P_2 , and hence, from (5), since the same law of winding is followed, that the cross section of the innermost layer of wire in the two instruments must be inversely proportional to P_1 and P_2 .

To have the same heating-error in the two cases, with outside resistance-coils of the same shape and size, it follows from (21) that it is necessary to use the same ratio between the resistance of the outside coil and the resistance of the magnetizing coil. To have a less or greater heating-error in the second than in the first case for the maximum deflection, it is only necessary to determine from (22) (the constants of which have been deduced by experiment with the first instrument) the greater or less volume V of the outside

resistance-coil needed with the second instrument, and calculate the ratio y of the resistance of the outside coil to that of the magnetizing coil by (21).

IX. As to the second error, arising from the sensibility of the instrument being temporarily varied by external magnetic disturbance referred to in § I. This can be overcome in three distinct ways : by

1. Shielding the instrument by putting it in an iron case with very thick walls.

2. Using a very powerful controlling field.

3. Using a motion of *translation*, and *not of rotation*, for the body in the instrument that is deflected by the current.

The first method to be successful requires that the iron case should be heavy and cumbersome, and makes it unsuitable for portable instruments.

A very powerful controlling field necessitates a very powerful deflecting force, and so either makes the instrument unsensitive or necessitates a considerable expenditure of electric energy in the instrument. The third method, consisting in giving the deflected body a *motion of translation*, is the best, since, no matter how strong the disturbing magnetic field may be, provided it is uniform throughout the small space traversed by the deflected body on its motion, no increase nor diminution of the deflection will be caused by the outside magnetic field. For this reason we have adopted this third method in our Magnifying Spring-instruments, and experiment shows that the shielding produced in this way is even better than in our older form of instruments provided with a powerful permanent magnet.

X. As to the third error, arising from certain types of voltmeters altering in sensibility so that their constants vary from time to time. This error arises from two causes. In some voltmeters, such as those in which a controlling permanent magnet is employed, there is an error which gradually comes on from the demagnetization of this magnet, an error which is greatly increased by a mistaken careful regular employment of a keeper for the magnet when the instrument is

not in use. This error can be avoided by dispensing with a permanent magnet, and using a spring or a weight, or an electromagnet which is magnetized by the current to be measured. A second time-error arises from the employment of iron in the neighbourhood, and has the effect of making the indications of the instrument depend not merely on the current passing at the time of measurement, but also on the currents that have lately passed through the instrument. This error is especially marked in those instruments in which the controlling force, or the force tending to bring the needle back to zero in opposition to the deflecting force, is produced by the attraction of an electromagnet containing a core of soft iron wire, and which it is *assumed* is magnetized to saturation by the current which it is our object to measure. With small currents, however, which produce deflections in the lower part of the scale, this iron core is not saturated, and differences in the values of the readings of as much as 30 to 40 per cent. are obtained, depending on whether it is an increasing or a decreasing current that is being measured with such an electromagnetic instrument.

XI. The fourth error, arising from the volt standards employed by various people differing several per cent. from one another, has been hitherto a very serious matter. The standards employed have usually been—

1. The E.M.F. of a Daniell's cell.
2. The E.M.F. of a Latimer Clark's cell.
3. A standard depending on the electrochemical equivalent of silver and the value of the legal ohm.
4. The indication of a voltmeter graduated by some well-known maker, but probably graduated in a way quite unknown to the user.

Standard 1 is vague unless the details of the method employed in setting up the Daniell's cell are more carefully specified than heretofore; for example, Carpart, in some recent tests of the E.M.F. of a Daniell's cell, made out of a U-tube, with a contraction in the lower part, found it to vary from 1.111 legal volts when the zinc-sulphate solution had 25 per cent. of ZnSO_4 , to 1.142 legal volts when the solution

had only 3 per cent. of ZnSO_4 . His method of testing is based on the electrochemical equivalent of silver and the legal ohm, and as a proof of its accuracy he finds that the E.M.F. of a Latimer Clark's cell is 1.434 volts. If, however, we compare his results for a Daniell's cell with 1.072 volts as found by Sir W. Thomson when the zinc-sulphate solution had a specific gravity of 1.2, we see how Daniell's cells differ among one another*.

The E.M.F. of a Latimer Clark's cell has also some vagueness attached to it, unless the temperature-coefficient of variation of E.M.F. of the particular Latimer Clark's cell is known and taken account of. Generally the users of these cells are contented to say that the E.M.F. of a Latimer Clark's cell, as determined by Lord Rayleigh, is 1.435 true volts, or 1.438 legal volts, and they neglect the temperature variation altogether, which amounts to about 1 per cent. diminution per 12°C . elevation of temperature with certain Latimer Clark's cells.

Standard No. 3 is the one that we have used for some time; and as we have obtained very satisfactory results with it, it may be well to describe our mode of using it a little in detail. First, by means of a large number of direct comparisons with a copper voltameter, we determine the most probable value in amperes of the indications in different parts of the scale of an ammeter *fixed permanently in position* by being screwed to the table. A current is now sent through this ammeter in series with a long thick copper wire having about 6 ohms resistance and wound in a large open coil. This coil has attached to its two ends the standard voltmeter, which is also *fixed permanently in position*. Simultaneous readings of the ammeter and voltmeter are now taken by two observers, when a signal is given and the bridge-piece of a mercury commutator is moved so as to cut off the current and connect the copper wire together with the voltmeter,

* Since the reading of this paper this subject has been very fully treated by Dr. Fleming, in a paper "On the Use of Daniell's Cell as a Standard of Electromotive Force," read before the Physical Society on June 27, 1885 [Phil. Mag. [5] vol. xx. p. 126 (August 1885)].

which is joined with it in parallel, to a differential galvanometer. Then *time* measurements, for about ten minutes, are taken by a third observer of the parallel resistance of the thick copper wire and the attached voltmeter. Next a curve is drawn connecting the resistance of the cooling coil with the time, and on completing the curve backwards we ascertain with great accuracy, by interpolation, the resistance at the moment of stopping the current, that is at the moment of making the simultaneous observations on the ammeter and voltmeter. Our copper wire is sufficiently thick that for a current of some 15 amperes passing through it, corresponding with about 80 volts maintained at its ends, the time-curve of its resistance, obtained after stopping the main current, is nearly a straight line with hardly any inclination to the axis along which the time is measured; or, in other words, the change of resistance due to cooling while the measurements of resistance are being made is very small and easily allowed for. And this resistance, in legal ohms, multiplied by the current passing in amperes, gives of course the potential difference in legal volts.

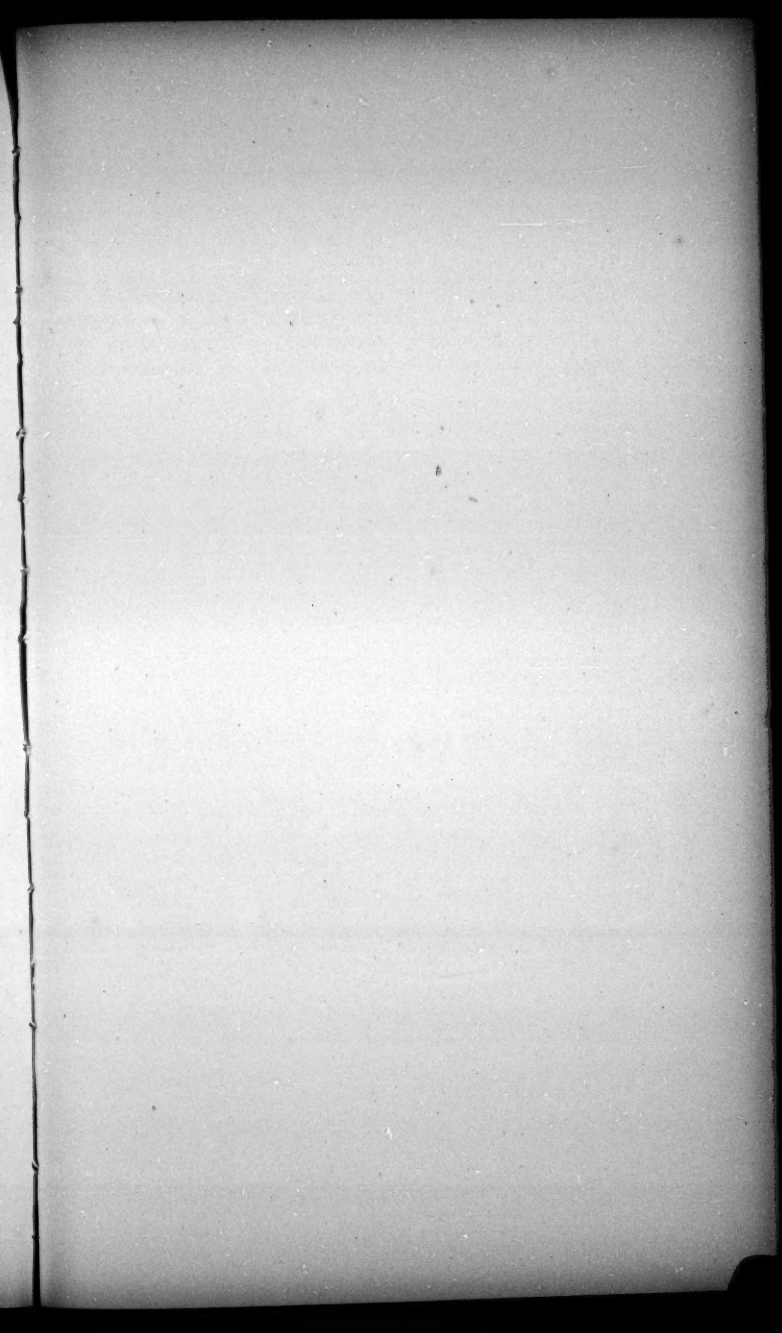
In using the copper voltameter to calibrate the ammeter it is of course necessary, in order that a firmly adherent deposit should be obtained, that the area of the immersed surfaces of the anode and cathode opposed to one another should exceed the limit given by Dr. Hammerl, viz. one square decimetre per 7 amperes, or rather more than two square inches per ampere; and in our experiments we have taken care to use a larger area in proportion to the current. Our negative plates consisted of a number of separate sheets of very thin hard copper, so that the weight of the deposit could be accurately measured; the current was supplied by accumulators and increased or diminished by a carbon resistance worked with a screw. By watching the ammeter and slightly turning this screw one way or the other, the current could be kept very constant for a long time. The value of the electrochemical equivalent employed for copper was 0.3295 milligrammes per coulomb, which is deduced from the mean of the electrochemical equivalents of silver, 1.118 milligrammes per coulomb as given by Lord Rayleigh, and 1.1183 by F. and W. Kohl-

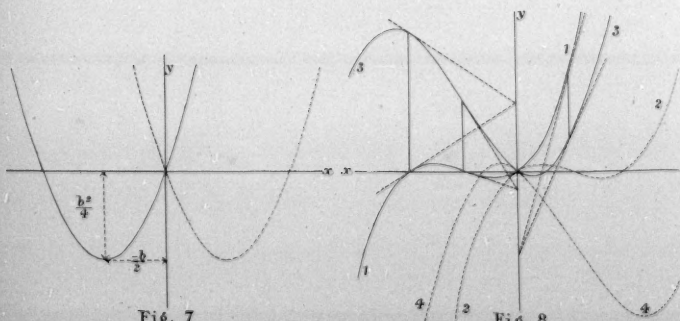
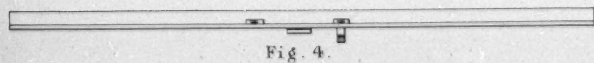
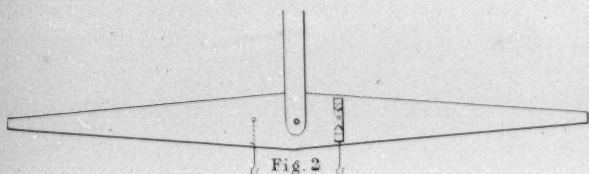
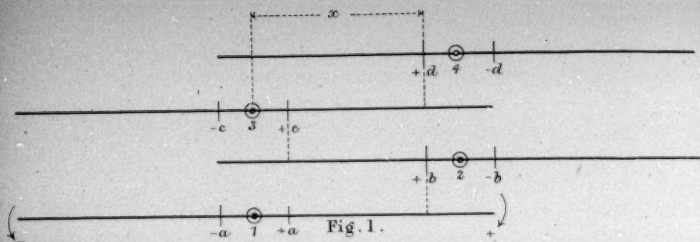
rausch, and by using 63.47, 107.56, 32.07, and 16 as the atomic weights of copper, silver, sulphur, and oxygen respectively. The solution of copper sulphate employed was a saturated one, the crystals of copper sulphate used in making it being chemically pure.

To avoid errors being introduced by oxidation of the copper that had just been deposited on the copper plates, they were, on being taken out of the sulphate of copper bath, on the stoppage of the current, immediately and quickly washed, first in water, secondly in alcohol, thirdly with ether poured over them, and then dried.

The current that a given set of accumulators will send through a copper voltameter depends of course partly on the distance between the alternate positive and negative plates, and partly on the depths to which they are immersed in the saturated solution of copper sulphate. The proper arrangements of the plates to allow any particular current to pass cannot conveniently be determined with the plates that are going to be used for the experiment, as an unknown weight of copper would be deposited on them during the process of adjustment, and great delay would be introduced if the plates had to be dried and weighed after the adjustment and before making the final experiment. Hence after the proper plates to be used as the anodes had been very carefully cleaned and weighed, a spare set of exactly the same size and shape were put into the bath, and their distance, as well as the carbon resistance, adjusted until the desired current was passing; then the spare set was removed and the weighed set substituted for them, and the experiment carried out.

Having determined our volt-standard in the way above explained, we tested various voltmeters of different makers, and we found that whereas the indications of some of them agreed very well with our standard, the indications of others differed by several per cent.





XXXVII. *On a Machine for Solving Equations.**By C. V. BOYS*.*

[Plate V.]

At a late meeting of the Physical Society Mr. C. H. Hinton showed some apparatus that he had made to explain the nature of equations. One of the instruments for representing the case of a simple equation was manageable enough; but the second, which dealt with the quadratic, could hardly be employed to solve a quadratic, as the root was not capable of continuous variation. It could, therefore, only after considerable trouble in trials and errors had been taken, find those real roots which consisted of whole numbers between the limits of the machine, say + and -10.

At the time that Mr. Hinton was trying to devise a quadratic machine (with the object not so much of solving quadratic equations, as of assisting boys to understand the method employed, now about ten years ago), he showed me what he had done; and it then occurred to me that continuous variation might be employed in a machine so that fractional roots could be found, and that the degree need not be limited to a quadratic, but that a cubic or one of any power of the form $a + bx + cx^2 + dx^3 + \&c. = 0$ could be solved. I then half completed the machine I have now the honour of exhibiting before the Physical Society, but laid it on one side till the present time.

I find that Mr. A. B. Kempe has invented and described† a machine with the same object. He replaces the expression x by $\cos \theta$, and then by a known process changes the equation in which the terms are powers of cosines of θ into one in which the terms are cosines of multiples of θ . The equation is then in a state to be dealt with by his machine. A series of levers jointed on one another, the first hinging on the intersection of a pair of rectangular axes, have their lengths variable, so that each one in order can be set so as to be equal to the coefficient of the corresponding term; moreover, by mechanism each one makes with the last the same angle that

* Read December 12, 1885.

† 'Messenger of Mathematics,' 1873, vol. ii. p. 51.

the first makes with the axis of x . Thus, on rotating the first, the end of the last describes a curved line cutting at one or more points the axis of y ; every angle θ of the first which causes this intersection is noted, and the value being inserted in the original equation gives at once one of the real roots.

I am not aware of any other machines for solving equations of a high order, except Mr. Cunynghame's, which will be described this afternoon. Machines for solving simultaneous equations also exist.

Like Mr. Hinton, I made use of the equality of moments by arranging a series of levers to operate upon one another. Thus, let the levers be called successively 1, 2, 3, 4, &c.; then 1 is on a stationary axis; it has at unit distance from its axis and on either side of it pivots, from each of which hangs a pan or hook labelled $-a$ and $+a$. If, then, a weight of a units is put upon either of these pans, the moment of the force upon the beam will be $-$ or $+$ a units.

Let a second beam 2 be connected with 1 by a sliding joint which is permanently at unit distance from the axis of 2; let the joint also carry a scale-pan, and let there be another scale-pan at unit distance on the other side. These are labelled $+b$ and $-b$. A weight of b units hung on either of these pans will produce a turning moment on b of $+$ or $-b$ units, and a turning moment on a of $+$ or $-bx$ units, where $x+1$ units is the distance between the axes of 1 and 2. Such a pair of beams of course will solve a simple equation whether the root be positive, negative, whole, or fractional; for as the second beam is made to traverse, it must pass some position where $a+bx=0$, and on the two sides of this position the moments on 1 will be opposite in sign. Therefore by shifting the beam till the arm changes position, the root can be exhibited by a pointer on a scale.

Similarly, if a third beam be mounted opposite 1 with a fixed axis, and if it be provided with pans as before, labelled $-c$ and $+c$, and if this be connected with 2, as 2 is with 1, then on pushing 2 along between these two beams a weight of c units on one of the c pans will exert on 2 a moment of $\mp cx$, and on 1 of $\mp cx^2$ units. If, then, a quadratic equation of the form $a+bx+cx^2=0$ has to be solved, weights of a , b , and c units are placed on the proper pans and the beam 2 slid

along until the beam 1 shifts its position. The two places at which this occurs are the two real roots of the quadratic. When the quadratic has no real roots, the machine can be still employed to find the impossible roots, as will be explained later.

A fourth beam placed opposite to 2 and connected with 3, as 2 is with 1, will in the same way make the machine capable of finding the real roots of a cubic equation, and so by adding more beams to the stationary and movable sets, an equation of any degree can in theory be solved. Of course, as the number of joints increases, the friction and elasticity of the working parts increase enormously, and so with the best work possible there is a practical limit beyond which it is hardly possible to go.

The only difficulty lay in the design of the beams, which would allow the connections between them to be capable of continuous variations at all parts of their length. The joint must be able to slide freely past the axis and past the coefficient pans. All the beams are of the same construction except that alternate ones have the connecting joint on opposite sides of the centre, as shown by the diagram fig. 1, Pl. V. Fig 2 is a front view of a beam with its supporting-rod and coefficient hooks complete; fig. 3 a back view, and fig. 4 a plan of the same. Fig. 5 is a transverse section through the middle, and fig. 6 an end view of a beam and connecting joint but without supporting bar. It will be seen that there runs along the back of each beam a web which may be embraced at any part by a pair of fingers fixed to the front of the next beam behind it, *i. e.* by the beam which is next higher in order of number.

It is evident that this construction allows the connecting joint of each beam to slide past the centre of the next lower one.

The supporting bars hang from a series of longitudinal beams lying in a cradle, the alternate ones being joined together. These then can be shifted relatively past one another, and one set can carry an index reading on a scale on the other set, so placed as to read zero when the distance of the centres of consecutive beams is one unit.

To show how to apply the machine to find the impossible roots of a quadratic, it will be necessary to consider the quadratic equation $y = a + bx + cx^2$ as representing a parabola. In

this parabola a has nothing to do with the shape of the curve. Variation in a only shifts it or makes it slide up and down the axis of y . It may therefore be struck out for the present. Then the parabola passes of necessity through the intersection of the axis of x and y . If the sign of b is changed, the new parabola will be a reflection of the old about the axis of y . Fig. 7 thus represents the double equation $x^2 \pm bx = y$.

The distances of the vertices of these parabolas from the axis of y are $\mp \frac{b}{2}$, and the values of y at the vertices are each $-\frac{b^2}{4}$. Now let a line parallel to the axis of x slide down the axis of y , and let its distance above the axis of x be a in a series of equations $a + bx + (c)x^2 = y$ (the c may be omitted), then those values of x at which this line cuts the parabola will be roots of the equations. As it slides down, that is as a increases algebraically, it approaches the vertex, and the two roots become more nearly equal until at the vertex they join, and there are two equal roots; here of course $b^2 = 4a$. If the sliding line is below this position, it misses the parabola, and there are no real roots.

Now going back to the case of real roots, it is clear that they are equally greater and less than $\mp \frac{b}{2}$, which is the distance of the vertex of the parabola from the axis of y , and the excess or defect is the half width of the parabola at the place of cutting. Now this half width is by the nature of the parabola equal to the square root of the distance along the axis from the vertex to the sliding line. If, then, the sliding line misses the parabola, the half width of the curve at the position of the line being equal to the square root of its distance above the vertex, will, as this distance is now negative, be impossible, or the curve at this region does not exist; nevertheless the imaginary half width is still the square root of the distance of the sliding line above the vertex. Geometry is incapable of representing this impossibility; but it is curious that the machine can be made to indicate its existence and measure its amount. Since the ordinates of the parabola represent the moments on the first beam for every value of x , it is clear that a minimum of moment will

be produced in the case of impossible roots. If, then, the machine fails to find two real roots (which of course must be within its limits), the first beam must be constrained by a spring instead of being free. The deflection will then indicate moments, and therefore the position of the minimum may be observed. The reading on the scale of the instrument is then the real part of the root, and the square root of the moment, which may be found by applying weights, the impossible part.

In the case of a cubic equation there must be one real root, and there may be three; if only one exists this may be found and divided out, when the resulting impossible quadratic can be solved by the machine. If all these roots are real, they can of course be found directly.

In a biquadratic, or one of a higher order, the possible roots may be found. If there remain more than two impossible roots, the machine is incapable of finding them.

While attempting to find properties of such equations that might be made use of in the construction of a machine, I found a curious relationship between two pairs of curves which might be turned to account in a curve-tracing machine. They are probably well known, but if so it may be worth while to mention them, as they have an interest in connection with this paper.

If the signs of all the terms of an equation $bx + cx^2 + dx^3 + \&c. = y$ be changed, the curve of course is in no way changed. If the signs of the even powers be changed, the curve is obviously turned round about the origin through two right angles; but if the sign of the term containing x alone be reversed, the shape of the curve is entirely altered, yet the relation between the new and the old curve is very simple.

Firstly, they cross at the origin, making there equal angles on either side of the axis of y . Secondly, whatever value of x be taken the tangents of the two curves for that value will cross one another and the axis of y in the same point. This must be so, for the distance of this point from the origin $= y - x \frac{dy}{dx}$, and in this expression no part of the term involving x remains. Fig. 8 shows the series of curves for the particular series of equations

$$y = x^3 + 3x^2 + 2x \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$y = x^3 - 3x^2 + 2x \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$y = x^3 + 3x^2 - 2x \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$y = x^3 - 3x^2 - 2x \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and shows the intersection of three pairs of tangents in y .

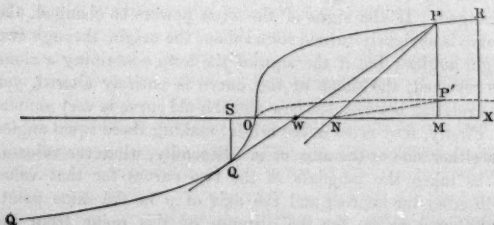
I should say, in conclusion, that I have brought this machine before the notice of the Physical Society not as a thing which is likely to be of any practical use, but because it has a certain interest, more especially in connection with another machine invented by my friend Mr. Cunynghame, which is really practically valuable, and which he will show this afternoon. (See below.)

XXXVIII. *On a Mechanical Method of Solving Quadratic and Cubic Equations, whether the Roots be real or impossible.*

By HENRY CUNYNGHAME, *Barrister-at-Law*.*

THE method which I have the honour to bring before this Meeting depends upon the use of a parabola of the form represented by the equation $x^n=y$. And it is capable of effecting solutions of all equations of the form $x^n+mx=C$.

Let us first examine the case of a cubic equation. By Cardan's rule reduce the equation to the form $x^3 + Ax + B = 0$. In fig. 1 let POQ be a cubical parabola, such that the ordi-



nate at any point represents the cube root of the abscissa along the axis of X, measured negatively towards the right. That is to say, let $OM = (PM)^3$. From P draw any line such

* Read December 12, 1885.

that the angle $PNM = \cot^{-1} A$, and let $ON = B$. Then it is clear that

$$OM + MN + ON = 0;$$

that is,

$$(PM)^3 + PM \cot PNM + ON = 0;$$

or, putting $PM = x$, $\cot PNM = A$ and $ON = B$, we have

$$x^3 + Ax + B = 0.$$

And this, therefore, shows that if from a point N , such that $ON = B$, we draw a line NP inclined at such an angle with OX that $\tan PNM = A$, then the value PM will represent a root of the equation $x^3 + Ax + B = 0$. Of course, if PN produced cuts the other branch of the cubical parabola, there will be two other real roots. If it does not, those other roots will be impossible.

In making a machine practically, it is well for convenience to draw on a paper a cubical parabola, say 2 feet wide, and make 1000 divisions on OX each way. Then make $XR = \sqrt[3]{1000} = 10$. But as this would make the parabola too thin, let us multiply its ordinates by 10, and then take a cotangent protractor scale with the cotangents also multiplied by 10. So that the parabola drawn, as well as the tangent scale in any position, is an orthographic projection of the real one. Such a machine will find real roots true to two places of decimals.

To find the impossible roots. First find the real root, say P ; draw PQ to touch the other branch of the parabola in Q . It may then be easily shown (though I am not aware that it has been before pointed out) that

$$SQ = \frac{PM}{2};$$

and that therefore

$$OS = SQ^3 = \frac{PM^3}{8}.$$

And, further,

$$\frac{SW}{MW} = \frac{SQ}{PM} = \frac{1}{2},$$

and

$$SW + MW = SO + OM = \frac{PM^3}{8} + PM^3 = \frac{9}{8} PM^3,$$

whence

$$3SW = \frac{8}{3} PM^3;$$

and

$$SW = \frac{8}{3} PM^3,$$

and

$$MW = \frac{6}{5} PM^3.$$

Now if α be the real root of the equation, so that $PM = \alpha$; then, by dividing $(x - \alpha)$ into $x^3 + Ax + B = 0$, we can easily show that the other two imaginary (or real) roots are given by the expressions

$$-\frac{\alpha}{2} \pm \sqrt{-\frac{3}{4}\alpha^2 + A},$$

B being $= \alpha^3 + A\alpha$. But we have already shown that

$$\frac{3}{4}\alpha^3 = \frac{8}{3} PM^3 = MW,$$

and we know that $A = \frac{MN}{MP}$; wherefore

$$\sqrt{A - \frac{3}{4}\alpha^2} = \sqrt{\frac{MN}{MP} - \frac{3}{4}\frac{MW}{MP}} = \sqrt{\frac{WN}{MP}};$$

which gives us this curious result—that if PM be the real root of the equation, the other two roots are

$$-\frac{PM}{2} \pm \sqrt{\text{ratio of } WM : PM}.$$

If W lies to the left of N , then WN is negative and the roots are impossible. If it lies to the right of N then the roots are real; and of course the ratio WN to PM is easy to find with the tangent scale; for it is the difference of the cotangents of the angles PNM and PWN , and may be read off at once by sliding the cotangent scale so as to have its centre at W and its edge parallel to NP ; or, in fact, in any convenient manner.

It remains only to add, that a geometrical relation can always be found for the impossible parts of the roots of all equations given by the form $x^n + mx = C$. Thus, in a parabola the impossible part would be represented by the horizontal distance of the line PN , measured along the axis of X from the tangent to the curve drawn *parallel* to PN , and would thus be the square root of a line, not of a ratio.

The same instrument may be used to take square roots of

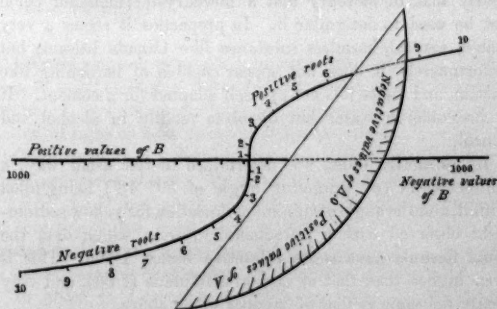
any numbers, so as to find the square root of the ratio above mentioned, by holding the cotangent protractor so that its centre is at O, and the division on its scale corresponding to the square of which the root is required is on the axis of X.

Then we shall have $\frac{OM}{PM} = \cot \alpha = \text{division on the scale}$.
But $OM = PM^2$; $\therefore$ division on the scale shows $\frac{PM^3}{PM}$, that

is to say PM^2 , whence PM gives the square root required. The values of PM should be written off in divisions and decimals along the branches OP and OQ, one being negative, the other positive; and the sides of the protractor, and also the axes OX, OX', should be marked with their proper signs.

It will be observed throughout that I have treated lines and areas and ratios of lines as representing numerical values. Hence the equations above given have numerical values for x ; and to make all the terms of the equations of the same order, constants would have to be introduced, which I have left out because they are unnecessary.

The following is a sketch of the machine represented in the act of solving the equation $x^2 - 20x - 200 = 0$.



(One of these instruments is to be seen at the loan collection, South Kensington, where it will remain permanently.)

XXXIX. *Note on some Organic Substances of High Refractive Power.* By H. G. MADAN, M.A.*

IN the course of some correspondence respecting M. Bertrand's polarizing-prisms, one of which I had the honour of showing to the Society last summer, I was informed that the cement used was naphthyl-phenyl-ketone dibromide. It was evident, from the theory of the prism, that this substance must have a refractive index at least as high as that of calc-spar for the ordinary ray, viz. 1.658; and as so highly refractive a cement seemed likely to be of great value for many purposes, I thought it worth while to prepare some of the ketone and examine its properties. I now exhibit a specimen of it.

I prepared it by the general method devised by Grucarevic and Merz (*Berichte der deutschen chemischen Gesellschaft*, vi. pp. 60 & 1238) for obtaining double ketones; viz. by heating together benzoyl chloride and naphthalene, and placing in the mixture a strip of zinc. The product was fractionally distilled, to free it from the excess of naphthalene, and was finally obtained as a thick yellow oil, boiling at a point so nearly that of mercury that a mercury-thermometer could not be used to determine it. In properties it seems a very stable, neutral, harmless substance like Canada balsam; but unfortunately it does not appear capable of hardening like balsam, and hence it is not by itself adapted for a cement. It is insoluble in water, but dissolves readily in alcohol and benzol.

Its refractive index was determined in the usual way, a hollow prism (of refracting angle of $59^{\circ} 48'$) being filled with it, and the angle of minimum deviation for yellow sodium-light observed with a refractometer; from which data the usual formula gave as its refractive index, 1.666. This is even higher than that of carbon disulphide (1.63), and very nearly the same as that of calc-spar given above.

I compared its dispersive power with that of a prism of very dense glass (refractive index 1.73) having a refracting angle of 60° , with the following results:—

* Read January 23, 1886.

Fraunhofer's line.	Ketone. $\mu =$	Glass. $\mu =$	Carbon disul- phide. $\mu =$
B . . .	1.654	1.725	1.617
E . . .	1.678	1.744	1.643

From this it is easily seen that the coefficient of dispersion of the ketone is $1\frac{1}{4}$ times that of the glass, and almost exactly the same as that of carbon disulphide.

I have made the compound of the ketone with bromine, referred to by M. Bertrand, but I hesitate to recommend it as a cement, at any rate for anything made of calspar; as it seems liable (like many similar bromides) to decompose with formation of hydrobromic acid, which of course acts upon the spar.

The ketone itself has a refractive index quite sufficiently high, if only some means can be found of hardening it.

I cannot, however, yet vouch for its permanency. It was made in August last, and remained for several months without change; but it has lately shown a tendency to pass from the colloidal to a crystalline condition, especially during the cold weather, although I had previously exposed it to a temperature of -20° without causing any alteration beyond an increase of viscosity.

I would also ask permission to show the Society a specimen of another highly refractive substance, viz., metacinnamene. It is a polymeric form of cinnamene, obtained by the action of light or heat upon the latter substance.

Cinnamene is a colourless liquid, obtained by distillation from the resin storax. Its refractive index I found to be 1.54 for the D line, nearly the same as that of Canada balsam. When this liquid is exposed to light for a few weeks, or heated to 190° in a sealed tube for half an hour, it becomes a glass-like solid, contracting greatly during the change.

In order to determine its refractive index, it was heated until it became viscous and placed in a hollow prism. The angle of minimum deviation for yellow sodium-light was found to be $39^{\circ} 44' 35''$; whence its refractive index is calculated to be 1.593. This is much higher than that of any other resin, so far as I can make out; and metacinnamene

would make a very valuable cement, since it is readily softened by heat and becomes remarkably hard and tough on cooling, but it does not show any great adhesiveness for glass. I hope to get good results by mixing it with other things, such as the ketone above described.

Another highly refractive organic substance is monobromonaphthalene, of which I exhibit a specimen.

It is a nearly colourless liquid, more stable than similar compounds containing bromine usually are, and boiling at the high temperature of 285°C . Its refractive index for the D line is 1.662; very nearly equal to that of naphthyl-phenylketone. I am rather surprised that it has not come into use as a substitute for carbon disulphide for filling prisms, as it is much less volatile and inflammable than the latter substance, while it has an even higher refractive and dispersive power.

The great desideratum at present is, a substance which has all the excellent qualities of Canada balsam—colourless, neutral, permanent in the air, becoming fluid when moderately heated, but hard and tough when cold, and with a refractive index of at least 1.66.

Such a substance would not only be of great use in the construction of polarizing and other prisms, but it would also be invaluable as a medium for mounting microscopic objects; since details of structure are brought out much more clearly when the object is immersed in a medium which differs greatly from it in refractivity. Phosphorus, arsenic sulphide, and mercury-potassium iodide have been used for this purpose, but they are all open to grave objections. None of them are permanent in the air; some are dangerously inflammable; while most of them act on delicate organic structures, although available for such things as siliceous Diatomaceæ.

The most hopeful direction in which to look is undoubtedly towards some of those complex organic compounds which are now being built up by many workers in England and Germany.

XL. *Note on the Determination of the Volume of Mercury in a Thermometer.* By A. W. CLAYDEN, M.A., F.G.S., Bath College*.

IN Professor Clark's paper on the Heat-capacity of a Thermometer†, he shows that it may be calculated approximately if the specific gravity of the instrument, and that of the glass of which it is made, be known. The first step in his method is to determine the volume of the contained mercury, which he points out may be ascertained by the following formula,

$$V_1 = \frac{V(S - S_2)}{S_1 - S_2}; \quad \dots \dots (1)$$

in which V_1 = the volume of mercury,

S_1 = its specific gravity,

V = the volume of the instrument,

S = its specific gravity,

S_2 = the specific gravity of the glass.

Now this expression will only be correct if the bulb and tube of the thermometer be completely filled with mercury, as they will be at a temperature (t°) slightly above the highest reading of the scale. At all lower temperatures V will be too large and S too small; an error being introduced, as Professor Clark points out, by the empty space of unknown capacity which occupies the unfilled portion of the tube.

At this higher temperature (t), V_1 becomes $V_1(1 + \beta t)$, β being the absolute coefficient of expansion of mercury. Similarly V becomes $V(1 + \alpha t)$, α being the coefficient of cubical expansion of the glass.

Also

S becomes $\frac{S}{1 + \alpha t}$; S_1 becomes $\frac{S_1}{1 + \beta t}$; and S_2 becomes $\frac{S}{1 + \alpha t}$.

If, now, these results be substituted for those in formula (1), the expression obtained will be theoretically correct, as the empty space vanishes at t° .

* Read January 23, 1886.

† Proc. Physical Soc. vol. vii. p. 113. [Phil. Mag. vol. xx. p. 48.]

Thus

$$V_1(1+\beta t) = \frac{V(1+\alpha t) \left(\frac{S}{1+\alpha t} - \frac{S_2}{1+\alpha t} \right)}{\frac{S_1}{1+\beta t} - \frac{S_2}{1+\alpha t}},$$

which, by simplification, becomes

$$V_1 = \frac{V(S-S_2)}{S_1 - \frac{S_2(1+\beta t)}{1+\alpha t}} \dots \dots \dots (2)$$

It will be seen that (2) gives rather a larger result than (1), because β is greater than α . This, however, is what must be expected if reference is made to (1). There V is too large and S is too small, so that the product VS remains unaltered; because $S = \frac{W}{V}$ if V is the volume in cubic centimetres, whence $VS = W$; but the product VS_2 becomes greater as V increases. Hence (1) gives too *small* a value for V_1 , a result which is compensated for in (2).

XLI. *Note on the Paper on some Thermodynamical Relations by Prof. W. Ramsay and Dr. S. Young. By Professors W. E. AYRTON, F.R.S., and JOHN PERRY, F.R.S.**

WE have written this note partly for the purpose of drawing attention to the fact that, although scientific work of many kinds is being much more thoroughly done now than ever before in the history of the world, mainly because the workers are specialists, yet a great deal of valuable work is rendered comparatively useless because the specialist is now too perfect a specialist, and so obtains no help from other departments of physics than that in which he himself is working.

We have ourselves, for example, wasted much time in useless observation of what seemed to be capricious behaviour of voltaic cells, a result to be expected when electricians who are not practical chemists conduct chemical investigations. Again, the specialist in pure mathematics might often do

* Read January 23, 1886.

much more valuable work if he had practised the application of his mathematics to natural phenomena, for he would learn of new directions in which to conduct his special investigations. We have been told by certain authorities in Chemistry that a few chemists with a good working knowledge of mathematics possess at the present time facilities for rapidly improving our knowledge of chemical science.

Drs. Ramsay and Young must forgive us for making their recent papers the text of a discourse of this kind. We might have selected papers by other authors which show more clearly the truth of what we say; but the work done by these two gentlemen is of such a valuable kind, their scientific positions are so well established, that we can venture to take this liberty without feeling that we do any harm in making an illustration.

These gentlemen read a paper at the last meeting of the Physical Society, in which they gave the gist of two papers, one of which had been published in the December number of the Philosophical Magazine, and the other has since been published in the January number. Members of the Society will find the first of these in the number of the 'Proceedings' of our Society which reached them yesterday. Referring to the papers in the Philosophical Magazine and to the published abstract of the paper read before the Physical Society, we have for saturated vapours:—

I. Ramsay's first law:

$\frac{L}{s_1 - s_2}$ is constant for all substances at the same pressure; L being latent heat, and s_2 and s_1 the specific volumes of the liquid and saturated vapour.

II. Ramsay's second law:

$\frac{L}{s_1 - s_2}$ at pressure p_1 bears a constant ratio to $\frac{L}{s_1 - s_2}$ at pressure p_2 for all substances.

III. Young's law:

$t \frac{dp}{dt}$ is constant for all substances at any given pressure; t being the absolute temperature, and p the pressure of a saturated vapour.

Their paper, published in the December number of the Philosophical Magazine and in our 'Proceedings,' is devoted

to the illustration of these three laws, and it is the record of an extraordinarily large amount of calculation.

IV. Ramsay and Young's law :

$\frac{d}{dp} \left(t \frac{dp}{dt} \right)$ is nearly constant for all stable substances at the same pressure.

One of us pointed out in the discussion on the paper, that to measure $\frac{dp}{dt}$ by drawing a tangent to a plotted curve was usually an inexact method ; and that much greater accuracy would have been obtained by the authors if they had, for each substance, found the values of the constants α , β , and γ in Rankine's well-known formula

$$\log p = \alpha - \frac{\beta}{t} - \frac{\gamma}{t^2} + \&c.; \quad . \quad . \quad . \quad (1)$$

and that probably the best way of making a comparison between two substances and observing changes due to dissociation would be by remarking that, for the substances examined by Regnault,

$$\alpha - \frac{\beta}{t} - \frac{\gamma}{t^2} + \&c. = \alpha - \frac{b}{\theta} - \frac{c}{\theta^2} + \&c., \quad . \quad . \quad . \quad (2)$$

if t and θ are the absolute temperatures of two saturated vapours at the same pressure. It is unlikely that any person will discover an improvement on Rankine's formula just given. It cannot merely be looked upon as an empirical formula being based on Rankine's molecular theory. The exactness with which Rankine has reproduced Regnault's numbers, using only three constants, is exceedingly remarkable. Rankine in his paper refers to the formula

$$\log p = \alpha - \frac{\beta}{t}, \quad . \quad . \quad . \quad (3)$$

which is the same formula but with two constants only, and is inexact for calculations of pressure.

To show that Messrs. Ramsay and Young might have employed a less laborious method of testing, we may first point out that it would have reduced the labour by something like 75 per cent. to have recognized the fact that I., II., III., and IV. are all identical. If any one of them is true, they must

all be true. Thus III., when stated mathematically, is simply

$$t \frac{dp}{dt} = \phi(p), \quad \dots \dots \dots (4)$$

$\phi(p)$ being the same function of the pressure for all substances. Differentiating (4) with regard to p , we obtain at once law IV. If it had sooner been observed that

$$\frac{L}{s_1 - s_2} = \frac{t}{J} \frac{dp}{dt} \quad \dots \dots \dots (5)$$

is simply a statement of the Second law of Thermodynamics, the authors would probably have examined the truth of the law in the shape III. only; since the specific volumes of vapours are so difficult to test experimentally, that Rankine, to obtain the density of steam, calculated $s_1 - s_2$ by dividing the latent heat L by $\frac{t}{J} \frac{dp}{dt}$ rather than depend on direct measurement. Again, since the first law is $\frac{L}{s_1 - s_2} = \phi(p)$, if we insert L_1 and L_2 , p_1 and p_2 , s and σ , we see that law II. is identical with I., and therefore I., II., III., and IV. are merely different methods of stating the same law, and to test one is to test them all.

Now we are sorry to say that we should have not attempted to test any of these forms of the law. The form which may be tested with most accuracy is III., or, as it may be stated mathematically, as in (4)

$$t \frac{dp}{dt} = \phi(p),$$

or

$$\frac{dp}{\phi(p)} = \frac{dt}{t},$$

or

$$t = a\psi(p), \quad \dots \dots \dots (6)$$

where t is the absolute temperature corresponding to the pressure p of a saturated vapour; a is a number which depends on the nature of the substance, and the function $\psi(p)$ is the same for all substances. Hence to test laws I., II., III., and IV. it is simply necessary to see whether (6) is true, as (6) is identical with them. Now if (6) is true, it follows that the ratio of the temperatures of two vapours to one another at

any pressure is the same as at any other pressure, or

$$\theta = kt. \quad \dots \dots \dots (7)$$

In fact, then, we can test laws I., II., III., and IV. by simply testing (7); and, as we find from Regnault's or Rankine's formula (1) or (2) that (7) is untrue, there is nothing further to be said about the four laws in question.

We draw attention to the fact that Dalton's law, which was the earliest law concerning saturated vapours, and of whose inaccuracy the text-books always warn us, is a much more correct law than the four identical laws which we have been considering. For, to state Dalton's law mathematically,

$$\frac{dp}{dt} = \phi(p). \quad \dots \dots \dots (8)$$

That is, the rate of change of pressure with temperature is constant at a given pressure for all saturated vapours. Integrating (8), we obtain Dalton's law in the shapes

$$t + a = \psi(p), \quad \dots \dots \dots (9)$$

$$\theta = t + b. \quad \dots \dots \dots (10)$$

That is, there is a constant difference between the temperatures of two saturated vapours at the same pressure. Now this law has been known to be incorrect for a long time; but if, assuming this law to be correct, we calculate the temperatures of saturation (corresponding to various pressures) of, say, bisulphide of carbon from those of water, we shall find our errors to be much smaller than those obtained when we use any of the above-mentioned four laws.

Of the law connecting θ and t , which has been published in the January number of the Philosophical Magazine, it is easy to see that the two forms in which it has been stated by the authors are inconsistent with one another. Such inconsistency is not of much importance in the statement of empirical laws; but we think that one of the forms is preferable to the other on account of its symmetry, as it may be put in the form:—The reciprocals of the absolute temperatures of two saturated vapours at corresponding pressures are linear functions of one another, or

$$\frac{m}{t} + \frac{n}{\theta} = 1.$$

Or, again, it may be put into the shape

$$\phi(p) = \alpha - \frac{\beta}{t} = \alpha - \frac{b}{\theta},$$

which may be compared with (3).

In conclusion, it will be observed that in calculating the temperature of a saturated vapour, a small-looking error may represent an error which looks much larger in the pressure. In comparing the errors of calculation from other empirical formulæ with those due to Rankine's formula, it is well to remember that Rankine calculated pressure, and that his errors, which for that reason might have been expected to appear large, were very small, thus showing the great accuracy of his formula.

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INDEX.

A.

	Page
Abney, W. de W., the production of monochromatic light, or of a mixture of colours, on the screen	181
Æther, on the structure of mechanical models illustrating some properties of the	74
Alums, on the specific refraction and dispersion of light by the	194
Anemometer, on an integrating	7
Ayrton, W. E., and Perry, John, note on the paper on some thermodynamical relations by Prof. W. Ramsay and Dr. S. Young ..	368
—, —, on the most economical potential-difference to employ with incandescent lamps.....	40
—, —, on the winding of voltmeters	334

B.

Baily, Walter, on an integrating anemometer	7
Bidwell, Shelford, on the generation of electric currents by sulphur-cells	256
—, on the sensitiveness of selenium to light, and the development of a similar property in sulphur	129
Binocular glasses adjustable to eyes having unequal focal lengths ..	80
Boys, C. V., on a machine for solving equations	355
Buchanan, John, on the thermoelectric position of carbon	185

C.

Calculating machine, summary of lecture on	81
— surface-tensions from the dimensions of flat drops and bubbles, on the error involved in Prof. Quincke's method of	145
Calibration of galvanometers by a constant current, note on the	285
Carbon, on the thermoelectric position of	185
Characteristic curves and surfaces of incandescence lamps	55
Chrono-barometer and chrono-thermometer, on a	85

	Page
Clark, J. W., on certain cases of electrolytic decomposition	118
—, on the determination of the heat-capacity of a thermometer . .	113
Clayden, A. W., on the determination of the volume of mercury in a thermometer	367
Cleminshaw, E., lecture-experiments on spectrum analysis	51
Comparison of the standard resistance-coils of the British Association with mercury standards constructed by M. J. R. Benoit of Paris and Herr Strecker of Würzburg	201
Cubic equations, on a mechanical method of solving quadratic and, whether the roots be real or impossible	360
Cunynghame, H., on a mechanical method of solving quadratic and cubic equations, whether the roots be real or impossible	360
Curves and surfaces of incandescence lamps, on the characteristic . .	55

D.

Daniell's cell, on the use of, as a standard of electromotive force . . .	161
Decomposition, on certain cases of electrolytic	118
Determination of the heat-capacity of a thermometer	113
— of the volume of mercury in a thermometer, note on the	367
Dispersion of light by the alums, on the specific refraction and . . .	194
Distribution of electric currents in networks of conductors treated by the method of Maxwell, problems in the	215
Dynamo, on the law of the electromagnet and the law of the	265
—, on the self-regulation of the compound	92

E.

Economical potential-difference to employ with incandescent lamps . .	40
Edmondson, Joseph, summary of lecture on calculating machines . .	81
Electric currents in networks of conductors treated by the method of Maxwell, problems on the distribution of	215
— — by sulphur-cells, on the generation of	258
Electrolytic decomposition, on certain cases of	118
Electromagnet, on the law of the, and the law of the dynamo	265
Electrometer, on the quadrant	7
Electromotive force of certain tin cells, a note on the	275
— —, on the use of Daniell's cell as a standard of	161
Equations, on a machine for solving	355
Error involved in Prof. Quincke's method of calculating surface-tensions from the dimensions of flat drops and bubbles	145
Eyes having unequal focal lengths, on binocular glasses adjustable to	80

F.

Fitzgerald, George Francis, on the structure of mechanical models illustrating some properties of the æther	74
---	----

	Page
Fleming, J. A., on the characteristic curves and surfaces of incandescence lamps	55
—, on the use of Daniell's cell as a standard of electromotive force	161
—, on molecular shadows in incandescence lamps	178
—, problems on the distribution of electric currents in networks of conductors treated by the method of Maxwell	215
Focal lengths, on binocular glasses adjustable to eyes having unequal	80
Fog, on the theory of illumination in a	87
Forms of polarizing-prism, on some new	39
Freezing-point of mercury, note on the verification of thermometers at the	283

G.

Galvanometers, note on the calibration of, by a constant current	285
Generation of electric currents by sulphur-cells, on the	256
Gladstone, J. H., on the specific refraction and dispersion of light by the alums	194
Glasses adjustable to eyes having unequal focal lengths, on binocular	80
Glazebrook, R. T., on a comparison of the standard resistance-coils of the British Association with mercury standards constructed by M. J. R. Benoit of Paris and Herr Strecker of Würzburg	201
Gray, J. Macfarlane, on the second law of thermodynamics	51

H.

Heat-capacity of a thermometer, on the determination of the	113
Herroun, E. F., note on the electromotive force of certain tin cells	275
Hopkinson, J., on the quadrant electrometer	7

I.

Illumination in a fog, on the theory of	87
Incandescence lamps, on molecular shadows in	178
—, on the characteristic curves and surfaces of	55
Incandescent lamps, the most economical potential-difference to employ with	40
Integrating anemometer, on an	7

L.

Lamps, the most economical potential-difference to employ with incandescent	40
—, on the characteristic curves and surfaces of incandescence	55
—, on molecular shadows in incandescence	178
Law of the electromagnet and the law of the dynamo	265
Lecture-experiments on spectrum analysis	51
Lecture on calculating machines, summary of	81
Light, on the sensitiveness of selenium to	129

	Page
Light, on the specific refraction and dispersion of, by the alums	194
—, the production of monochromatic, or of a mixture of colours, on the screen	181
Line-divider, the uses of a	1

M.

McConnel, James C., notes on the use of Nicol's prism	22
Machine for solving equations, on a	355
Madan, H. G., on some new forms of polarizing-prism	30
—, on some organic substances of high refractive power	364
Malcolm, Col., on binocular glasses adjustable to eyes having unequal focal lengths	80
Marks, Miss Sarah, on the uses of a line-divider	1
Mather, T., note on the calibration of galvanometers by a constant current	285
Mechanical method of solving quadratic and cubic equations, whether the roots be real or impossible	360
— models illustrating some properties of the æther, on the structure of	74
Mercury in a thermometer, on the determination of the volume of . .	367
— standards from Paris and Würzburg, a comparison of the standard resistance-coils of the British Association with	201
Molecular shadows in incandescence lamps	178
Monochromatic light, the production of, or of a mixture of colours, on the screen	181
— telescope, with application to photometry, on a	90

N.

New forms of polarizing-prism, on some	39
Nicol's prism, notes on the use of	22
Note on some organic substances of high refractive power	364
— on the calibration of galvanometers by a constant current	285
— on the determination of the volume of mercury in a thermometer .	367
— on the electromotive force of certain tin cells	275
— on the paper on some thermodynamical relations by Prof. W. Ramsay and Dr. S. Young	368
— on the verification of thermometers at the freezing-point of mercury	283
Notes on the use of Nicol's prism	22

O.

Organic substances of high refractive power, note on some	364
---	-----

P.

Perry, John, and Ayrton, W. E., note on the paper on some thermodynamical relations by Prof W. Ramsay and Dr. S. Young	368
--	-----

	Page
Perry, John, and Ayrton, W. E., on the most economical potential-difference to employ with incandescent lamps.....	40
—, —, on the winding of voltmeters	334
Photometry, a monochromatic telescope, with application to	90
Polarizing-prism, on some new forms of	39
Potential-difference to employ with incandescent lamps, the most economical	40
Prism, notes on the use of Nicol's.....	22
Problems on the distribution of electric currents in networks of conductors treated by the method of Maxwell	215
Production of monochromatic light, or of a mixture of colours, on the screen	181
Properties of the æther, on the structure of mechanical models illustrating some	74

Q.

Quadrant electrometer, on the	7
Quadratic and cubic equations, on a mechanical method of solving, whether the roots be real or impossible	360
Quincke's method of calculating surface-tensions from the dimensions of flat drops and bubbles, on the error involved in	145

R.

Ramsay, W., and Young, S., on some thermodynamical relations	289, 307, 327
Rayleigh, Lord, on a monochromatic telescope, with application to photometry	90
—, on the theory of illumination in a fog	87
Refraction and dispersion of light by the alums, on the specific	194
Rücker, A. W., on the self-regulation of the compound dynamo....	92

S.

Second law of thermodynamics, on the	51
Self-regulation of the compound dynamo, on the	92
Sensitiveness of selenium to light, and the development of a similar property in sulphur, on the	129
Shadows in incandescence lamps, on molecular	178
Solving equations, on a machine for.....	355
— quadratic and cubic equations, whether the roots be real or impossible, on a mechanical method for	360
Specific refraction and dispersion of light by the alums.....	194
Spectrum analysis, lecture-experiments on	51
Standard resistance-coils of the British Association with mercury standards from Paris and Würzburg, a comparison of the.....	201
Stanley, W. Ford, on a chrono-barometer and chrono-thermometer	85

	Page
Structure of mechanical models illustrating some properties of the ether.....	74
Substances of high refractive power, note on some organic	364
Sulphur-cells, on the generation of electric currents by.....	258

T.

Telescope, on a monochromatic, with application to photometry....	90
Theory of illumination in a fog, on the	87
Thermodynamical relations, on some	289, 307, 327
Thermodynamics, on the second law of	51
Thermoelectric position of carbon, on the	185
Thermometer, on the determination of the heat-capacity of a	113
—, determination of the volume of mercury in a	367
Thermometers at the freezing-point of mercury, note on the veri- fication of	283
Thompson, S. P., on the law of the electromagnet and the law of the dynamo.....	265
Tin cells, a note on the electromotive force of certain	275

U.

Use of Daniell's cell as a standard of electromotive force	161
— of Nicol's prism, notes on the	22
Uses of a line-divider, on the.....	1

V.

Verification of thermometers at the freezing-point of mercury, note on the	283
Voltmeters, on the winding of	334
Volume of mercury in a thermometer, note on the determination of the.....	367

W.

Whipple, G. M., note on the verification of thermometers at the freezing-point of mercury	283
Winding of voltmeters, on the	334
Worthington, A. M., on the error involved in Prof. Quincke's method of calculating surface-tensions from the dimensions of flat drops and bubbles	145

Y.

Young, S., and Ramsay, W., on some thermodynamical relations ..	289
---	-----

PROCEEDINGS
AT THE
MEETINGS OF THE PHYSICAL SOCIETY
OF LONDON.

SESSION 1885-86.

February 14th, 1885.

Dr. F. GUTHRIE, President, in the Chair.

The following was elected a Member of the Society :—

Prof. GEORGE FULLER.

The following communications were made :—

"On a Line and Area Divider." By Miss SARAH MARKS.

"On Recent Additions to his Integrating Anemometer." By Mr.
WALTER BAILY.

February 28th, 1885.

Dr. F. GUTHRIE, President, in the Chair.

The following were elected Members of the Society :—

Mr. OSBERT CHADWICK, late R.E. ; Mr. G. R. BEGLEY.

The following communications were made:—

"Notes on the Use of Nicol's Prisms." By Mr. J. C. McCONNEL.

"On Polarizing Prisms." By Mr. H. G. MADAN.

"On the most Economic Potential-difference to employ with Incandescent Lamps." By Profs. AYRTON and PERRY.

"On the Second Law of Thermodynamics." By Mr. J. MAC-PARLANE GRAY.

March 14th, 1885.

Dr. F. GUTHRIE, President, in the Chair.

The following communications were made:—

"On Recent Researches in Radiation." By Capt. ABNEY.

"On the Characteristic Curves of Incandescence Lamps." By Dr. J. A. FLEMING.

"On Lecture-Experiments in Spectrum Analysis." By Mr. E. CLEMINSHAW.

"On Sir Wm. Thomson's Quadrant-Electrometer." By Dr. J. HOPKINSON.

March 28th, 1885.

Dr. F. GUTHRIE, President, in the Chair.

The following was elected a Member of the Society:—

Mr. F. B. HAWES.

The following communications were made:—

"On Calculating Machines, illustrated by a Collection of Ancient and Modern Machines, lent by the owners for the occasion." By Mr. JOSEPH EDMONDSON.

"On the Structure of Mechanical Models illustrating some of the Properties of the *Æther*." By Prof. G. F. FITZGERALD.

Instruments and Apparatus were exhibited by Col. E. D. MALCOLM, Mr. W. F. STANLEY, Mr. CONRAD C. COOKE, Mr. HILGER, and Dr. W. M. WATTS.

April 25th, 1885.

Dr. F. GUTHRIE, President, in the Chair.

The following communications were made:—

"On the Theory of Illumination in a Fog." By Lord RAYLEIGH.

"On a Monochromatic Telescope, with Applications to Photometry." By Lord RAYLEIGH.

"On the Self-Regulation of the Compound Dynamo." By Prof. A. W. RÜCKER.

"On the Determination of the Heat Capacity of Thermometers." By Mr. J. W. CLARK.

"On Mr. Sugg's Inclined Photometer." By Mr. LIVINGSTONE.

May 9th, 1885.

The Meeting was held in the Physical Lecture Room, Clifton College, Bristol.

Dr. F. GUTHRIE, President, in the Chair.

The following were elected Members of the Society:—

Mr. E. F. HERBORN; Mr. ARTHUR L. SELBY, B.A.;

Mr. E. CLEMINSHAW, M.A.

The following communications were made:—

"On Evaporation and Dissociation." By Prof. W. RAMSAY and Dr. S. YOUNG.

"On a Model illustrating Propagation of the Electro-magnetic Wave." By Prof. S. P. THOMPSON.

"On a Self-recording Stress and Strain Indicator." By Prof. H. S. HELE SHAW.

"On the so-called Silent Discharge in Ozone Generators." By Mr. W. A. SHENSTONE.

May 23rd, 1885.

Dr. F. GUTHRIE, President, in the Chair.

The following was elected a Member of the Society:—

Mr. A. H. FISON, D.Sc.

The following communications were made:—

"Experiments showing the Variations caused by Magnetization in the length of Iron, Steel, and Nickel Rods." By Mr. SHELFORD BIDWELL.

"On the Spectral Image produced by a Rotating Vacuum-Tube." By Mr. SHELFORD BIDWELL.

"On the Action of Light in Diminishing the Electrical Resistance of Selenium, and on a Method of producing similar effects with Sulphur." By Mr. SHELFORD BIDWELL.

"On certain cases of Electrolytic Decomposition." By Mr. J. W. CLARK.

"Notes on Electrical Symbols." By Mr. J. MONRO.

June 13th, 1885.

Dr. F. GUTHRIE, President, in the Chair.

The following communications were made:—

"On the Comparison of Mercury Standards of Resistance issued by M. Mascart with those of the British Association." By Mr. R. T. GLAZEBROOK.

"On the Winding of Voltmeters." By Profs. AYRTON and PERRY.

"On the Manner in which Light affects the Resistance of Selenium and Sulphur Cells." By Mr. SHELFORD BIDWELL.

"On the Error involved in Prof. Quincke's Method of Calculating Surface-Tensions from the Dimensions of Flat Drops and Bubbles." By Mr. A. M. WORTHINGTON.

June 27th, 1885.

Dr. F. GUTHRIE, President, in the Chair.

The following were elected Members of the Society:—

THOMAS HANDS, M.A.; FRANCIS HENRY NALDER; Prof. W. RAMSAY, Ph.D.; F. W. SANDERSON, B.A.; W. A. SHENSTONE.

The following communications were made:—

"On the Specific Refraction and Dispersion of the Alums." By Dr. J. H. GLADSTONE.

"On a Form of Standard Daniell Cell and its Application for Measuring Large Currents." By Dr. J. A. FLEMING.

"On the Phenomenon of Molecular Radiation in Incandescent Lamps." By Dr. J. A. FLEMING.

"On Problems in Networks of Conductors." By Dr. J. A. FLEMING.

"On Lecture-Experiments in Colour Mixtures." By Capt. ABNEY.

"On the Stream-Lines of Moving Vortex Rings." By Prof. O. J. LODGE.

"On the Thermoelectric Position of Carbon." By Mr. J. BUCHANAN.

"Further Experiments on Sulphur Cells." By Mr. SHELFORD BIDWELL.

November 14th, 1885.

Dr. F. GUTHRIE, President, in the Chair.

The following communications were made:—

"On Testing Thermometers at the Melting-point of Mercury." By Mr. G. M. WHIFFLE.

"On the Electromotive Force of certain Tin Cells." By Mr. E. T. HERROUN.

"On the Law of the Electromagnet, and the Law of the Dynamo." By Prof. S. P. THOMPSON.

November 28th, 1885.

Dr. F. GUTHRIE, President, in the Chair.

The following was elected a Member of the Society:—

THOMAS HOLMES BLAKESLEY, M.A., M.I.C.E.

The following communications were made:—

"On the Calibration of Galvanometers by a Constant Current." By Mr. T. MATHER.

"On a new Driving Clock-work of Isochronous Motion." By Mr. A. HILGER.

"On a new Direct-Vision Spectroscope." By Mr. A. HILGER.

"On a Machine for the Solution of Equations." By Mr. C. VERNON BOYS.

"On a Machine for the Solution of Cubic Equations." By Mr. H. H. CUNYNGHAME.

"On some Thermodynamic Relations." By Drs. RAMSAY and YOUNG.

December 12th, 1885.

Dr. F. GUTHRIE, President, in the Chair.

The following were elected Members of the Society :—

CHARLES FREDERIC CASELLA ; Prof. T. E. THORPE, F.R.S.

The following communications were made :—

“On a Magneto-electric Phenomenon.” By Mr. WYATT.

“On some Thermodynamic Relations.” By Drs. RAMSAY and YOUNG.

January 23rd, 1886.

Dr. F. GUTHRIE, President, in the Chair.

The following communications were made :—

“Note on a Paper by Prof. W. Ramsay and Dr. S. Young ‘On some Thermodynamic Relations.’” By Profs. AYRTON and PEREY.

“Note on a Paper by Mr. J. W. Clark on the Determination of the Heat-Capacity of a Thermometer.” By Mr. A. W. CLAYDEN.

“Note on some Organic Substances of High Refracting Power.” By Mr. H. G. MADAN.

“On an Apparatus consisting of a Vibrating Metal Strip with associated Air-Column, illustrating the Motion of Matter at the Nodes.” By the PRESIDENT.

Annual General Meeting.

February 13th, 1886.

Dr. F. GUTHRIE, President, in the Chair.

The following Report of the Council was read by the President :—

In presenting their Annual Report the Council are able to congratulate the Society on the steady position which the work of the Society has attained. On looking through the ‘Proceedings’ it will be seen that the first five volumes include the work of nine years; but that the sixth volume is entirely occupied with the work of one year, viz. the Session which commenced with the Ordinary Meeting held on February 9, 1884, and terminated with the Annual General Meeting held on February 14, 1885. The Council, feeling that it

was now possible to fix definitely the times at which the Parts of the 'Proceedings' should be issued to Members, have decided that these parts should be issued quarterly, viz. in July, October, January, and April.

It is intended that these four Parts should form one volume, so that in future each volume of 'Proceedings' should contain the work of one Session. The first three Parts of Vol. VII. have already been issued in accordance with this arrangement. Among the various papers of interest which have been presented to the Society, the Council desire to call special attention to the valuable paper by Prof. Rücker on the Compound Dynamo.

The Society held a Meeting last summer in Bristol, at which the Educational Institutions of the town were visited; and the Council desire to express their thanks to the Members and other friends residing at Bristol for the efficiency of the arrangements which were made.

In the Annual Report of last year it was mentioned that space had been allotted to the Physical Society in the International Exhibition of Inventions. Cases were lent by the Science and Art Department, and other cases were well filled by Apparatus lent by Members of the Society. Besides the Apparatus exhibited in the space allotted to the Physical Society, many things were shown by our Members in other parts of the Exhibition. The Council desire to record their thanks to the Council of the Inventions Exhibition for the allotment of space and for other facilities.

The hope expressed in last year's Report that the second volume of the reprint of Dr. Joule's papers would be issued in the current year has not been realized. More than half the volume, however, is in type, and Dr. Joule expects to be able to supply the rest of the copy at an early date; the Council therefore feel confident that they will be able to complete the book this year.

The Library now consists of 512 separate works; 220 Transactions of Societies, Proceedings, &c.; 130 separate Memoirs.

Many valuable gifts have been received from the various learned Societies and Authors at home and abroad. Amongst which may be mentioned:—

Journal of the Society of Arts.

Reports of the Natural History Society, Canada.

Catalogue of Canadian Plants. Part II. Gamopetalæ.

Clark's Transit Tables.

Proceedings of the Cambridge Philosophical Society.

Journal and Proceedings of the Royal Society of New South Wales.
 Proceedings of the Royal Society of London.

Institution of Mechanical Engineers, Proceedings.

Proceedings of the Royal Society of Edinburgh.

Abhandlungen des Universität zu Tōkiō.

Transactions of the Seismological Society of Japan.

Recherches sur les Lois des l'écoulement et du choc des Gaz.

G.-A. Hirn.

Chemical and Physical Journal, St. Petersburg.

Quarterly Journal of the Royal Meteorological Society.

Norwegian North-Sea Expedition. Zoologi. Crustacea: 1^A, 1^B.

Papers on Technical Education, Applied Science Buildings, Fittings,
 and Sanitation. E. C. Robins.

Electrical Review.

Observations and Researches made at the Hong Kong Observatory
 in 1884, by Dobereck.

Magnetical and Meteorological observations, Government Obser-
 vatory, Bombay.

Circulars of information, Bureau of Education, Washington.

Radcliffe Library, Oxford. Catalogue of Additions.

Mauritius Meteorological Observatory Reports.

The Council have considered that the Library might be made more useful to Members by the institution of a Librarian, who should be applied to in all matters relating to the Library, and they are glad to announce that Mr. Boys has kindly consented to perform the duties of this office.

The Council regret to announce the deaths of three Members—
 Prof. T. Andrews, Prof. J. W. Clark, and Mr. W. Ladd.

THOMAS ANDREWS was born at Belfast on the 19th December, 1813. After receiving his early education in his native town, he proceeded to Glasgow, where he studied Chemistry under Dr. Thomas Thomson; and thence to Trinity College, Dublin, where he distinguished himself both in Classics and Science. He continued his Chemical Studies in Paris under Dumas. In 1835 he obtained his Medical Degree in the University of Edinburgh; and returning to Belfast, in which he thenceforth spent his life, he gained a highly successful practice as a Medical man, and devoted his leisure time to scientific research. Dr. Andrews gave up his medical practice, and also the chemical lectureship in the Royal Belfast Academy, upon his appointment in 1845 to the vice-presidentship of Queen's College (or, as it was then called, "the Northern College"), Belfast.

The duty of advising the Government in maturing a plan for the Queen's Colleges and of getting them into working order fell upon the Presidents and Vice-Presidents, and in the performance of this duty Dr. Andrews took a prominent part. He was himself appointed Professor of Chemistry at Belfast, and held the office for 30 years. In 1849 he was elected a Fellow of the Royal Society. In 1852 he was President of the Chemical Section of the British Association at Belfast, and occupied the same position in Edinburgh in 1871. Five years later he was President of the Association at Glasgow. In 1879 he resigned his appointments at Queen's College, and led a very retired life until his death on November 26th, 1885. Dr. Andrews was remarkable for his originality as a scientific thinker, and for the extreme delicacy and care of his experimental work. His early work earned from the Royal Society one of the Royal Medals as far back as 1844; but the importance of his discoveries continually increased. One of his great investigations was the determination of the heat evolved during chemical action. He here led the way in a study, the great value of which is now thoroughly recognized, and his earlier papers on this subject were soon followed by those of Favre and Silberman. Another important paper was that communicated to the Royal Society on the subject of ozone, in which he showed by a series of remarkably delicate experiments that ozone was oxygen in an altered or allotropic condition. The study of ozone was continued by Andrews and Tait, and their paper on the subject published in the *Phil. Trans.* 1860 announced the discovery of its constitution.

The crowning work of Dr. Andrews's life was his research into the continuity of the Liquid and Gaseous states of Matter, which employed his highest powers of experiment, and which must remain to all time as one of the most marked steps in the progress of Physical Science.

WILLIAM LADD, who was one of the original Members of this Society, was born at Deal in 1815 and came to London at the age of 14. He took a prominent part in bringing Electricity into practical use, and was well known as the constructor of the largest induction-coils and electrical apparatus. In 1867 Mr. Ladd brought forward a dynamo machine, in which many of the difficulties in constructing dynamos for practical use were overcome. In 1878 he introduced from America the Wallace Farmer system of electric lighting, and effected in it considerable improvements. One of the earliest installations of electric lighting, viz. that at the Liverpool

Street Terminus of the Great Eastern Railway, was made under Mr. Ladd's supervision.

Mr. Ladd took an active part as director in the promotion and management of the Anglo-American Brush Electric Light Corporation (Limited), and also of the Electric Power Storage Company.

He died on April 16, 1885.

The Council are indebted to Dr. O. J. Lodge for the following memoir.

J. W. CLARK, the late Assistant Professor of Physics in University College, Liverpool, was the son of Mr. Joseph Turner Clark, of Southampton, member of the Society of Friends, and was born on 1st July, 1856. From his earliest childhood he is reported to have been studious, and to have taken many prizes in scientific subjects at Weston School, whither he went for two years, at the age of 15. His father destined him for business, but he managed to attend evening classes at the Hartley Institution, Southampton; and on the occasion of a lecture by Professor Barrett in that town he made earnest appeal to the professor to use his influence and help him towards a scientific career. He was permitted to accompany Mr. Barrett on a lecturing tour in the North, and afterwards to enter as a student in the Royal College of Science, Dublin, 1874 and 1875. Professor Barrett may be thus considered to have commenced and directed Clark's scientific development, and speaks of him as follows:—"Clark was the most enthusiastic student of science I have ever known. It was very difficult to get him away from the laboratory, or from his books, at night. His only fault was a tendency to discursive reading and a dislike to study those things for which he had no taste. He made a series of investigations when with me on the effect of high pressure and temperature on certain of the chemical and physical properties of liquids, and nearly blew his eye out by an explosion: he was at work again before his eye was healed."

Under Professor Barrett he acquired considerable manipulative dexterity and experimental facility, which were extended and widened by his next two years of study at Heidelberg. For both Professor Bunsen and Professor Quincke he retained the warmest attachment, and indeed veneration. The life in Germany deeply modified his whole tone of thought. He made many friends there, and became to all intents and purposes a German student.

The experience gained under Professors Quincke and Bunsen remained probably the dominant influence of Clark's life; and to

the end, *i. e.* some nine years after, he was still engaged in researches the germs of which had been planted at Heidelberg.

From Germany he proceeded, as Demonstrator in Chemistry and Physics, to the Royal Indian Engineering College at Coopers Hill, under Professor McLeod. While at Coopers Hill he continued several researches, and published some of them; but publication was his weak point, at least in the opinion of the writer. By the time the research was sufficiently complete for him to think of publication, the vivid interest of it seemed to have evaporated, or perhaps something else began to fill his mind, and accordingly what he wrote was somewhat perfunctorily put together.

The writer is convinced, however, that statements made in his papers may be relied on for most careful accuracy; and nothing vexed him more than to hear or see a statement of his loosely questioned or cavilled at. Yet the casual or parenthetical aspect of some of his statements, suggesting to a reader that they were perhaps mere hasty surmises, often led to criticisms of this sort; whereas the statement really embodied, and was intended to express and convey the gist of, the labour of weeks.

After some four years of work at Coopers Hill he resigned his post and lived at home for eighteen months; when, as the result of some scientific correspondence between him and the writer, he came to University College, Liverpool, in the Spring of 1883—first, rather informally, afterwards as official Demonstrator in Physics, and finally as Assistant Professor. Into the organization of the incipient Physical Laboratory, then going on, he threw himself heart and soul; working at all out-of-the-way hours, and far into, or even through, the night. These violent exertions of course told upon his health, and necessitated periods of quiescence. Often remonstrated with, he replied that he could not help it, that he must work furiously when the fit seized him; and so, indeed, it seemed.

Such absorbing devotion to experimental research the writer never before encountered; and no less remarkable were the ready ingenuity and intuitive skill with which difficulties were encountered and foreseen. A difficulty, or a lack of an experimental kind, instantly seemed to brighten up every faculty of the man's nature; and, sometimes at once, sometimes after a night's brooding, sketches of all manner of contrivances were thrown off, and the selected one soon put into execution.

Had he been a less reckless worker, and thus kept in better health, and had there been a place in England where a man's

bodily wants were provided for, leaving him free to pursue research without let or hindrance, he would have done great things; but there is no such place; though, indeed, the writer tried his very best to create such a place for him at Liverpool, and in this attempt was nobly seconded by the Principal and Council of the College: and perhaps if there had been, the health difficulty would have interfered. However that may be, he showed signs of breaking down at the end of the long vacation—which, it is needless to say, is only named “holiday” in the sense of being more deeply and continuously consecrated to the diviner portion of one’s work—and he was granted one term’s leave of absence for foreign travel. He returned to his old and well-loved haunts at Heidelberg, saw Prof. Quincke and the laboratory once more, and died there at the early age of 29, on the 27th of last October.

A sad end to a most promising career! His loss to the Physical department of the College is irremediable; the loss to science, though unfelt, is probably not less essentially severe. To the writer, the loss of his invigorating criticism, unique scientific culture, and strong personal friendship is a painful blow.

The Treasurer made the Financial Statement given below.

The Society then proceeded to the election of Officers and other Members of Council for the ensuing year. The Council was constituted as follows:—

President.—Prof. BALFOUR STEWART, LL.D., F.R.S.

Permanent Vice-Presidents.—Dr. J. H. GLADSTONE, F.R.S.; Prof. G. C. FOSTER, F.R.S.; Prof. W. G. ADAMS, M.A., F.R.S.; Sir WILLIAM THOMSON, D.C.L., LL.D., F.R.S.; Prof. R. B. CLIFTON, F.R.S.; Dr. F. GUTHRIE, F.R.S.

Other Vice-Presidents.—Prof. W. E. AYRTON, F.R.S.; SHELFORD BIDWELL, M.A., LL.B.; Prof. H. McLEOD, F.R.S.; Prof. W. CHANDLER ROBERTS-AUSTEN, F.R.S.

Secretaries.—Prof. A. W. REINOLD, M.A., F.R.S.; WALTER BAILY, M.A.

Treasurer.—Dr. E. ATKINSON.

Demonstrator.—C. VERNON BOYS.

Other Members of Council.—CONRAD W. COOKE; Prof. G. FORBES, F.R.S.E.; Prof. F. FULLER, M.A.; R. T. GLAZEBROOK, M.A., F.R.S.; J. HOPKINSON, D.Sc., F.R.S.; Prof. J. PERRY, F.R.S.; Prof. J. H. POYNTING, M.A.; A. W. RÜCKER, M.A., F.R.S.; Prof. S. P. THOMPSON, D.Sc.; C. ALDER WRIGHT, D.Sc., F.R.S.

Votes of thanks were passed to the Lords of the Committee of Council on Education; to the PRESIDENT and other OFFICERS of the Society; and to the AUDITORS.

PROPERTY ACCOUNT OF THE PHYSICAL SOCIETY.

ASSETS.		LIABILITIES.	
	£ s. d.		£ s. d.
Balance in Bank	116 18 6	Due to Treasurer	5 10 4
Estimated Subscriptions due	36 0 0	Subscriptions for 1886	4 0 0
£400 Furness 4 per cent. Debenture Stock	436 0 0	Balance	1678 8 2
£400 Midland 5 per cent. Preference Stock	648 0 0		
£200 Metropolitan Board of Works Stock	217 0 0		
£200 Lancaster Corporation Stock	214 0 0		
	<u>£1687 18 6</u>		<u>£1687 18 6</u>

We have examined the above Account, and also the Securities at the Bank, and find the same to be correct.

E. ROBERT FESTING, }
GEORGE FULLER, } Auditors.

London, February 1st, 1886.